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## Measuring the natural yield curve

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### Abstract

We generalize the concept of the natural rate of interest (Laubach and Williams, 2003; Woodford, 2003) by defining and estimating the natural yield curve (NYC) - the term structure of natural interest rates. Our motivation stems i.a. from the observation that at times when central banks attempt to directly affect long-term interest rates (e.g. via quantitative easing) the gap between the short-term real and natural rate is no more a good indicator of the monetary policy stance. We estimate the NYC on US data, document its main properties and show i.a. that in the period 2008-2011 the NYC allows to better capture the US monetary policy stance than the short-term natural rate.

**JEL:** C32, E43, E52,

**Keywords:** Natural rate of interest, natural yield curve, unconventional monetary policy, Kalman filter

# 1 Introduction

Since its introduction to economics in the late XIXth century (Wicksell, 1898), the natural rate of interest (NRI) has noticed up and downswings of popularity. The recent revival of interest in the NRI concept was related to the introduction of inflation targeting by several prominent central banks in the 1990's. In this strategy monetary authorities control the short-term interest rate in order to stabilize inflation at the targeted level. As a consequence the concept of an equilibrium level of the short-term interest rate that stabilizes the economy (and inflation) received much attention both from the theoretical (Woodford, 2003) and empirical (Laubach and Williams, 2003) points of view.

Despite its obvious attractiveness, the NRI also has some deficiencies. First, it is an unobservable variable, with all consequences for its measurement and uncertainty surrounding its estimates. While this does not diminish its appeal as a theoretical concept, it may undermine its practical usage by policymakers (Orphanides and Williams, 2007). Second, the NRI always provoked confusion as to what maturity it should be applied. For Wicksell the natural rate was a long-term concept (Amato, 2005). This notion can be also found in the contemporaneous literature, e.g. Bomfim (2001). However, most recent studies approach the NRI from the short-term perspective, in line with the character of the operational target of many central banks (e.g. Crespo Cuaresma et al., 2004; Garnier and Wilhelmsen, 2005; Brzoza-Brzezina, 2006; Mesonnier and Renne, 2007; Edge et al., 2008; Andrés et al., 2009). While ultimately, whether the NRI is treated as a long or short-term concept is simply a matter of definition, this duality does certainly not help the NRI's popularity and understanding. Finally, the developments following the financial crisis 2007-09 changed the way central banks look at their policy instruments. After hitting the zero lower bound on interest rates several central banks (e.g. the Fed, Bank of England, ECB) decided to engage in so called unconventional monetary policies. A primary example is quantitative easing introduced by the Fed. One important goal of this policy was to lower long-term interest rates. This means that central banks went beyond their traditional policy of affecting directly only the short end of the yield curve (with some exceptions like the Twist operation in the US in the 1960's). It should be noted here, that while central banks' impact on longer-term rates has been particularly pronounced since the outbreak of the financial crisis, even before policymakers were actively using communication tools in order to influence the yield curve (e.g. Blinder et al., 2008).

While obviously not much can be done against the first deficiency, our contribution is to deal with the latter two. To this end we generalize the NRI concept by defining a new unobservable variable - the natural yield curve (NYC). Our definition is a direct analogue to Laubach and Williams (2003) NRI definition - the gap between the real and natural yield curves determines the output gap, which in turn determines inflation. In particular, when the yield curve gap is closed, so becomes the output gap and inflation stabilizes. As a result we obtain the whole natural yield curve, which nests natural rates of various maturities. Even more importantly, the NYC (and the associated yield curve gap) provide a synthetic measure of policy restrictiveness, particularly useful at times when central banks attempt to directly affect the longer end of the yield curve. The latter is our main motivation to develop and estimate the NYC.

In addition to the natural rate research our study is directly related to two strands in the literature. The first is yield curve modeling. Our empirical approach must rely on a parsimonious representation of the (real) yield curve. A number of approaches has been developed that allow for modeling the yield curve as a function of a small number of parameters. One of them are affine models usually with the no-arbitrage restriction imposed. These models have been originally proposed by Duffie and Kan (1996) and further developed by Duffie (2002) (for the classification of the affine models see also Dai and Singleton, 2000). Due to the no-arbitrage restriction affine models have been mostly used for derivative pricing, however some studies applied them for analysing the relationships between macroeconomic variables and the yield curve as well (Rudebusch and Wu, 2008). The second group of models originates from the decomposition of the yield curve proposed by Nelson and Siegel (1987) and extended by Svensson (1994) and Christensen et al. (2009). The Nelson-Siegel approach allows to describe the yield curve with a small number of latent factors, which due to certain restrictions imposed on their loadings, can be interpreted as a long, medium and short-term factor (often called level, curvature and slope)<sup>1</sup>. Despite the absence of the no-arbitrage restriction these models match the yield curve quite well and are widely used by many institutions (i.a. central banks) for yield curve modeling (BIS, 2005). In our research we rely on the specification of Nelson and Siegel (1987). Not only does it model relatively well our real yield curves, but also, as will be explained later, greatly facilitates the identification of the NYC.

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<sup>1</sup>Svensson (1994) adds a second curvature factor to improve the goodness of fit and Christensen et al. (2009) incorporate a second slope factor to impose the no-arbitrage restriction.

The second strand are macro-financial models that link the yield curve to macroeconomic developments. These describe the joint dynamics of bond yields and macroeconomic variables i.a. by means of structural (e.g. Rudebusch and Wu, 2008) or VAR (e.g. Ang and Piazzesi, 2003, Diebold et al., 2006) models. The general conclusion is that relationships work in both ways - macroeconomic developments affect the yield curve, which in turn affects the macroeconomy. However, in contrast to our research, these models do not incorporate any latent macroeconomic variables, in particular no measures of equilibrium interest rates.

Our model is estimated on US data for the period 1q1983 - 2q2011. Main findings are as follows. First, our approach allows to estimate the natural yield curve. Second, the related yield curve gap matters for the output gap and inflation. Third, it shows features that make it potentially more valuable than the traditional short-term interest rate gap. In particular it is able to document the accommodative stance of monetary policy after the financial crisis.

The rest of the paper is structured as follows. Section 2 describes the data and presents the yield curve and macroeconomic models. Section 3 discusses the estimation and Section 4 concludes.

## 2 Model and data

Our modeling approach is based on two steps. First, we describe the yield curve as function of three latent factors.<sup>2</sup> Next, we use these factors as observables in our macroeconomic NYC model. This allows to estimate the latent macroeconomic variables - potential output and the natural yield curve. A two-step estimation procedure means that we loose on efficiency. However, the big number of latent variables (level, slope, curvature, the natural slope and potential output) as compared with a relatively small number of observable variables (interest rates, GDP and inflation) would make the one-step joint estimation very complicated numerically. Given this estimation procedure, we also describe the two models separately.

### 2.1 The Nelson-Siegel model

As mentioned in the Introduction, several methods of modeling the yield curve are available. We choose the Nelson-Siegel approach for two reasons. First, since our purpose is macroeconomic rather than financial market oriented, we are ready to trade off some precision in modeling the curve against simplicity of the model and the economic interpretation of the parameters. Second, as explained in section 2.2 the Nelson-Siegel model provides a natural and intuitive identifying restriction that allows us estimate the NYC. The yield curve is modeled as a function of three unobservable factors. The functional form is:

$$r_{\tau,t} = L_t + S_t \left( \frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) + C_t \left( \frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right) \quad (1)$$

where  $r_{\tau,t}$  is the spot interest rate of maturity  $\tau$ .  $L_t$ ,  $S_t$  and  $C_t$  are the level, slope and curvature factors respectively and  $\lambda$  is a constant that governs the exponential rate of decay of the loadings.

This model has been extensively documented in the literature and hence, we discuss it only briefly here. In order to give some intuition behind the way the Nelson-Siegel function works, in Figure 1 we present the loadings associated with the three factors. The loading on the first factor is constant. Hence, it affects all interest rates proportionally and so determines the curve's level. The loading on the second factor affects short rates more than long rates and hence determines

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<sup>2</sup>We decided not to add a second curvature factor as proposed by Svensson because it helps to fit the longer end of the yield curve while we use maturities up to 10 years only which can be matched by one curvature factor. Moreover it would be difficult to find a clear economic interpretation for this factor.

the slope of the yield curve. The last factor is hump-shaped which means that it exerts the largest impact on medium-term rates. As a consequence it allows to model the curvature of the yield curve.

Nelson and Siegel show that equation (1) is appropriate for modeling nominal yield curves. In particular it explains over 90% of their variability. In Section 3.1 we document that it performs well in modeling the real yield curve as well.

## 2.2 The natural yield curve model

We model the natural yield curve as an analogue to Laubach and Williams' modeling of the natural rate of interest. The core of the model are two standard macroeconomic equations, the IS curve and the Phillips curve. The former is:

$$x_t = A_x(L)x_{t-1} + A_G(L)GAP_{t-1} + \epsilon_{1t} \quad (2)$$

where  $x_t$  denotes the output gap (i.e.  $x_t \equiv y_t - y_t^*$ , where  $y_t$  and  $y_t^*$  stand for output and potential output respectively),  $GAP_t$  is the yield curve gap to be defined below,  $A_x(L)$  and  $A_G(L)$  are polynomials in the lag operator and  $\epsilon_{1t}$  is a serially uncorrelated error term.

The Phillips curve takes the form:

$$\pi_t = B_\pi(L)\pi_{t-1} + B_x(L)x_{t-1} + \epsilon_{2t} \quad (3)$$

where  $\pi_t$  denotes the inflation rate,  $B_\pi(L)$  and  $B_x(L)$  are polynomials in the lag operator and  $\epsilon_{2t}$  is a serially uncorrelated error term.

Further, we assume that potential output follows an autoregressive process:

$$y_t^* = y_0 + C_y(L)y_{t-1}^* + \epsilon_{3t} \quad (4)$$

where  $y_0$  is a constant,  $C_y(L)$  is a polynomial in the lag operator and  $\epsilon_{3t}$  is a serially uncorrelated error term. The assumed properties of the lag operators (whether they imply stationarity or unit roots in the underlying series) are discussed in the estimation section.

Finally, the definition of the yield curve gap must be given. We define  $GAP_t$  as the area between the real and natural yield curves. Following the discussion in section 2.1 we have:

$$\begin{aligned}
GAP_t &\equiv \int_0^T (r_\tau - r_\tau^*) d\tau \\
&= (L_t - L_t^*)T + (S_t - S_t^*) \int_0^T \frac{1 - e^{-\lambda\tau}}{\lambda\tau} d\tau + (C_t - C_t^*) \int_0^T \frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} d\tau
\end{aligned} \tag{5}$$

where  $r_\tau^*$  is the spot natural interest rate of maturity  $\tau$ ,  $L_t^*$ ,  $S_t^*$  and  $C_t^*$  are the level, slope and curvature of the NYC and  $T$  is the maturity horizon taken into account. Unfortunately, such a definition of the gap does not allow for the unique identification of  $L_t^*$ ,  $S_t^*$  and  $C_t^*$ . To see this note that for any given  $L_t$ ,  $S_t$  and  $C_t$  there is an infinite number of combinations of the natural level, slope and curvature that yield the same value of  $GAP_t$ . In other words, for any given real curve there is an infinite number of natural curves such that the area between the two remains the same. This means that we have to impose two additional identifying restrictions in order to uniquely pin down the NYC.

The first restriction is based on the assumption that in the very long run the instantaneous forward real rate equals its natural counterpart. In other words we assume that infinitely far in the future investors' best guess about the short-term real rate is its natural level. This can be written as:

$$\lim_{\tau \rightarrow \infty} r_{\tau,t}^f = \lim_{\tau \rightarrow \infty} r_{\tau,t}^{f*} \tag{6}$$

where  $r_{\tau,t}^f$  and  $r_{\tau,t}^{f*}$  are respectively the instantaneous real and natural forward rates. In the Nelson and Siegel (1987) model the forward curve can be written as:

$$r_{\tau,t}^f = L_t + S_t e^{-\lambda\tau} + C_t \tau e^{-\lambda\tau} \tag{7}$$

so that assumption (6) yields the restriction  $L_t = L_t^*$ . Here we see an important reason for choosing the Nelson-Siegel model for describing the yield curve - it provides us with a simple and intuitively appealing restriction that greatly facilitates the estimation of the NYC. Note that this restriction yields another interpretation of our identifying assumption - the central bank cannot influence very (infinitely) long-term real spot interest rates.<sup>3</sup>

The second restriction assumes that the curvature of the NYC is constant. This is in turn motivated by the fact, that in the literature macroeconomic

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<sup>3</sup>This follows the observation derived from (1) that  $L_t = \lim_{\tau \rightarrow \infty} r_\tau$

factors are found not to affect the curvature of the yield curve (see Diebold et al., 2006). However, the curvature could be affected by the central bank aiming for instance at flattening the yield curve after having hit the zero lower bound. If the curvature is affected rather by monetary policy than by macroeconomic developments we can assume that the curvature of the natural curve is constant over time.<sup>4</sup>

Finally, we have to assume a process for the evolution of our remaining latent variable  $S_t^*$ . Following the literature on the natural interest rate we assume that this variable follows an  $AR(1)$  process:

$$S_t^* = \rho_s S_{t-1}^* + \epsilon_{4t} \quad (8)$$

where  $\epsilon_{4t}$  is a serially uncorrelated error term. Thus we do not force the natural slope to be a random walk process. We allow for the more general  $AR(1)$  process and we test the level of integration later on.

## 2.3 Data

We estimate the model on the basis of quarterly US data starting in 3q1983 and ending in 2q2011 (114 observations). Nominal interest rates for various maturities have been drawn from the St. Louis FED database. We used the yields on T-Bills for maturities of 3 and 6 months and yields on zero-coupon and fixed-coupon Treasury Bonds for 1, 2, 3, 5, 7 and 10 years maturity. Due to substantial gaps in the data we skipped the longer-term maturities of 20 and 30 years. Hence, in our application  $T = 40$  (measured in quarters).

The purpose of our project is to estimate the real yield curve so we had to adjust the nominal interest rates for inflation expectations first. We used the estimates of inflation expectations for selected maturities provided by the Cleveland FED database as calculated by Haubrich et al. (2008).

As a measure of inflation we took the CPI core inflation measure calculated by the BLS (CPI inflation less food and energy). Using this index instead of CPI allows to adjust inflation for the shocks not directly related to the domestic output gap. The level of GDP at constant prices has been taken from the BEA database. Both macroeconomic variables: core CPI and GDP were seasonally adjusted.

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<sup>4</sup>Sometimes humps on the yield curve may result from the lack of liquidity or inefficiency of the market. This, however, does not seem to be the case for the US T-bonds market.

### 3 Estimation

As noted before, we conduct the estimation in two steps. In the first stage we estimate the yields-only dynamic Nelson-Siegel (DNS) model. From the DNS model we calculate the estimates of three factors, which can be interpreted as level, slope and curvature. In the second step we use these factors as the observable variables in the NYC model.

#### 3.1 The Nelson-Siegel model

The dynamic Nelson-Siegel model can be expressed in state space form. We follow Diebold et al. (2006) and estimate the DNS model simultaneously using the Kalman Filter. The alternative approach proposed by Diebold and Li (2002) relies on the estimation of the decay parameter  $\lambda$  with OLS for the cross-sectional data and then on the derivation of the level, slope and curvature with  $\lambda$  given from the first step as a mean or median calculated over time. Diebold et al. (2006) argue that the one-step method is superior to the two step approach.<sup>5</sup>

The block of measurement equations consists of the stochastic version of (1) for  $\tau = 1, 2, 4, 8, 12, 20, 28$  and 40:

$$r_{\tau,t} = L_t + S_t \left( \frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) + C_t \left( \frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right) + \varepsilon_{\tau,t} \quad (9)$$

The state equations form a VAR(1) process:

$$\begin{bmatrix} L_t - \mu_L \\ S_t - \mu_S \\ C_t - \mu_C \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} L_{t-1} - \mu_L \\ S_{t-1} - \mu_S \\ C_{t-1} - \mu_C \end{bmatrix} + \begin{bmatrix} \xi_{1t} \\ \xi_{2t} \\ \xi_{3t} \end{bmatrix} \quad (10)$$

where  $\mu_L$ ,  $\mu_S$  and  $\mu_C$  denote the means of the respective factors. The model can be written in compact form with measurement equations expressed as

$$r_t = \mathbf{A}f_t + \varepsilon_t \quad (11)$$

and the state equations as

$$(f_t - \mu) = \mathbf{A}(f_{t-1} - \mu) + \xi_t \quad (12)$$

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<sup>5</sup>Pooper (2007) reviews different strategies for estimation of DNS model examining both one-step and two-step approaches.

$$\text{where } f_t = \begin{bmatrix} L_t \\ S_t \\ C_t \end{bmatrix}, r_t = \begin{bmatrix} r_{1,t} & r_{2,t} & r_{4,t} & r_{8,t} & r_{12,t} & r_{20,t} & r_{28,t} & r_{40,t} \end{bmatrix}', \mu = \begin{bmatrix} \mu_L \\ \mu_S \\ \mu_C \end{bmatrix}.$$

We assume that error terms in the measurement equations are not cross-correlated, but allow for cross-correlation of the disturbances in the state equations. Moreover, we impose the restriction of no correlation between error terms from the measurement and state equations which is a common procedure in the estimation of the DNS model as a state space model (see Diebold et al., 2006). Hence:

$$\begin{pmatrix} \varepsilon_t \\ \xi_t \end{pmatrix} \sim WN \left[ \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{H} \end{pmatrix} \right],$$

where  $\mathbf{D}$  is a diagonal matrix.

We estimate the model consisting of (11) and (12) by Maximum Likelihood using the Berndt-Hall-Hausman (BHHH) algorithm with starting values drawn from Diebold et al. (2006).<sup>6</sup>

The results are collected in Table 1. The first row presents the mean values of subsequent factors as defined in (10). The mean of the level factor amounts to 0.0366. This value is very close to the sample mean of the real yield on 10Y bonds, which is an empirical counterpart for the level factor.<sup>7</sup> The mean of the slope factor is negative and slightly lower than the sample mean of the spread between 3-months and 10-years yields. The third factor is negative as well and lower than its empirical counterpart.

<sup>6</sup>The discussion on numerical issues related to the estimation and calibration of the DNS model can be found in Gilli et al. (2010)

<sup>7</sup>Following the literature (e.g. Diebold et al., 2006, Diebold and Li, 2002) we refer to empirical counterparts of the estimated factors. From (1) we have  $\lim_{\tau \rightarrow \infty} r_{\tau,t} = L_t$  and  $\lim_{\tau \rightarrow 0} r_{\tau,t} - \lim_{\tau \rightarrow \infty} r_{\tau,t} = S_t$ .

It follows that the closest empirical counterpart for the level factor is the longest available yield and for the slope factor the difference between the shortest and longest yield. For our estimate of  $\lambda = 0.2255$  these are:  $r_{40,t} = L_t + 0.11S_t + 0.11C_t$  and  $r_{1,t} - r_{40,t} = 0.78S_t - 0.01C_t$ . The empirical counterpart for the curvature factor is in turn defined as the difference between twice the medium-term yield and the sum of short- and long-term yields. This is  $2r_{8,t} - r_{1,t} - r_{40,t} = -0.08S_t + 0.39C_t$ .

These relations show that for the finite maturities the estimated slope and curvature factors would have higher amplitudes and variances than their empirical counterparts.

As far as the persistence of the factors is concerned, the first factor is the most persistent one with the autoregression coefficient equal to almost 0.99 (though significantly below one). The curvature is less persistent (0.92) and the slope even less so (0.72). The high persistence of the level factor and the lower persistence of slope and curvature is mostly in line with empirical evidence (Gasha et al., 2010). Some of the off-diagonal elements of the covariance matrix are statistically significant reflecting the contemporaneous relationship between factors.

The estimate of the decay parameter  $\lambda$  equals 0.2255 (with standard error of 0.002). This corresponds to 0.075 for monthly data, a number very close to the estimates reported by Diebold et al. (2006) for nominal yields (0.077).

Figures 2, 3 and 4 show the level, slope and curvature factors together with their empirical counterparts. The level factor and the real 10Y yield move together in a downward trend across the sample. Nevertheless, there are some periods when the yield on the 10-years bonds declined more than the level factor - in particular in the beginning of the 1990's, in the first half of the last decade and at the end of the sample (2010-2011). Still, the correlation between the two series is very high (0.98).

The correlation between the slope factor and its empirical counterpart is strong as well (0.99). The slope factor has slightly higher volatility and the amplitude than the spread (by construction, as pointed out in footnote 7) and the maxima and minima occur at the same points in time.

The lowest but still high correlation can be observed between the curvature factor and its empirical counterpart. The correlation coefficient equals to 0.98 and the curvature is also more volatile than the combination of empirical yields.

The reported correlations do not differ significantly from the corresponding correlations for nominal yields reported by Gasha et al. (2010), while Diebold et al. (2006) even indicate lower correlations for slope and curvature (for the sample ending in 2000). Our conclusion is that the DNS model is suitable for modeling the US real yield curve.

### 3.2 The NYC model

In the second step we estimate the natural yield curve model expressed by (2), (3), (4) and (8). The fitted slope and curvature from the DNS model derived in the previous step are used as observable variables. Again, it is convenient to put the model in state space form and to use the Kalman Filter for estimation. We use four observable variables: inflation, output, slope and curvature and two

latent variables: natural slope and potential output. As mentioned above we assume that the natural level of the real yield curve equals its current level and that the natural curvature is constant. Hence, the only “natural” factor which has to be estimated in this step is the natural slope.

In the measurement equations we specify the number of lags of dependent variables on empirical basis. We find that two lags of both, the output gap and inflation are sufficient to capture the dynamics of these variables. Moreover we impose the restriction on the coefficients for lagged inflation in the Phillips curve to sum to one, implying a unit root in the inflation series. The yield curve gap in the IS curve and the output gap in the Phillips curve enter both with one lag.

As far as the state equations are concerned we assume that potential output follows an  $AR(2)$  process imposing again a unit root via the restriction on the autoregressive parameters to sum to one. The natural slope is assumed to follow an  $AR(1)$  process and the error terms from all the equations are assumed to be neither auto- nor cross-correlated.

Regarding the “pile-up problem” raised by Stock and Watson (1998) we calibrate the variance for potential output and natural slope, while other parameters are estimated by Maximum Likelihood using the BHHH algorithm. Choosing the variance for the error term in the potential output equation we used the estimates of potential output calculated by the CBO as a benchmark. We did not have any benchmark for the natural slope so we set the variance in the natural slope equation slightly below the variance of the error term in the analogous equation for the empirical slope derived from the DNS model.

Tables 2 and 3 collect the estimation results for the NYC model. The yield curve gap proves statistically significant, entering the IS equation with a negative sign. It means that if the real yield curve remains below the natural one the output gap increases.

The sum of the parameters for the own lags of the output gap is significantly below one (the hypothesis that both coefficients sum to one has been rejected by the data at the usual significance level).

As far as the Phillips curve is concerned the output gap lagged by one period enters the equation with a positive sign, in line with economic theory. The output gap is statistically significant at usual significance levels.

Analysing the state equations we see that the autoregressive coefficient in the natural slope equation takes a value significantly below one. With the Wald test statistic we reject the hypothesis that the natural slope follows a random walk. Moreover the ADF test for the smoothed values of natural slope strongly rejects

the unit root. The results for the potential output equation let us conclude that potential output is integrated of order one. The Wald test does not support the hypothesis of a second unit root in the potential output series.

Figure 5 presents smoothed values of the natural slope together with appropriate standard errors. According to the results the slope of the natural yield curve reaches its local maxima in 1989, 1998 and in 2006. Minima occur in 1992, 2002 and recently since 2008 lasting till the end of the sample.

Figure 6 shows the estimate of potential output and Figure 7 presents the comparison of the output gap calculated from the NYC model with the output gap published by the Congressional Budget Office (CBO). It can be seen that the NYC output gap matches CBO's estimates quite well with only two exceptions: the late 1990s, when the NYC model shows a smaller positive output gap and the recent global crisis, when the CBO negative output gap is deeper than the gap from the NYC model. In particular, the latter result can be explained by the fact that our output gap is identified via the Phillips curve on the basis of the behaviour of core inflation. Despite a deep and long lasting recession core inflation did not fall substantially due to exchange rate depreciation and the NYC model identified a smaller output gap.

In contrast to the output gap, there is no external benchmark for the yield curve gap. We document its relationship with macroeconomic variables on Figures 8 and 9. The former shows the estimates of the yield curve gap together with our output gap led by 8 quarters. The correlation between these variables is clearly visible and amounts to 0.46. The second figure shows the yield curve gap and core inflation led by 9 quarters. Here the correlation is slightly weaker but still significant and equals -0.26. All in all, our estimates seem to match economic theory.

The historical values of the natural yield curve have been presented on Figure 11.

## 4 The natural yield curve vs. the natural interest rate

In Section 3.1 we have shown our estimated natural yield curve, output gap and yield curve gap, and how they are related to macroeconomic variables. Now it is time to concentrate on the question whether estimating the whole natural yield curve improves upon the (possibly simpler) estimation of the short-term natural interest rate.

Using our estimates of the natural level, slope and curvature we can calculate the natural interest rate of any desired maturity. This, by itself, is an advantage of the NYC over the narrowly defined NRI. In particular we can calculate the natural level for the short-term (3-months) real rate, corresponding to the natural interest rate as defined by Laubach and Williams (2003).

On Figure 10 we present the estimates of the 3-months real rate gap (the difference between the 3-month real and natural rates) and the yield curve gap. Both indicators can be interpreted as alternative measures of the monetary policy stance. A positive value of the gap indicates restrictive monetary policy, a negative one corresponds to accommodative monetary policy.

The behaviour of the yield curve gap shows that monetary policy was expansionary in the years 1983-89, since 1992 till the end of the 1990s (with the exception for some quarters in 1994/95) - covering the dot-com bubble period, between 2003 and 2004 and since the mid-2009 till the sample end. Clearly restrictive monetary policy can be identified in the years 1989-91, 1994-95, the beginning of the last decade (2000-2002) and since 2005 till 2009.

As mentioned before, the crucial difference between both measures is that the short-term real rate gap does not take into account the longer end of the yield curve, which may be meaningful for economic agents in the decision making process. For the most of the sample the two estimates match relatively well. This is also confirmed by the relatively high correlation coefficient (0.57). Still, for some periods substantial divergence can be observed.

In particular at the end of the sample the short-term rate gap differs from the yield curve gap. According to the short-term rate gap since 3q2007 till the end of the sample monetary policy was restrictive. Due to the strongly negative output gap and low inflationary pressure the short-term real rate which would balance the economy should have been significantly lower than it was. As is well known, this could, however, not be achieved because of the zero lower bound

problem.

Instead of lowering the short term rate, the Fed launched the programme of quantitative easing (two phases) purchasing commercial papers and longer-term Treasury bonds. The programme allowed to lower the longer end of the yield curve decreasing the cost of long-term financing and in fact making monetary policy more accommodative. This effect can be observed from the yield curve gap turning negative since 3q2009. Thus, the measure covering the whole yield curve defines the monetary policy stance more appropriately than the short-term rate gap only, which indicates restrictive monetary policy in this period.

The advantage of looking at the yield curve gap instead of relying on the short-term rate gap only can be better understood by looking at Figure 12. This shows the natural and real yield curves for specific dates. For example in 3q2003 (panel a) the Fed managed to raise the policy rate above the natural short-term rate. However, the longer end of the yield curve was still below the natural one, so that monetary conditions remained accommodative, leading to an overheating of the economy. According to our model, monetary conditions became restrictive only in 2006-07 (panel b), when the yield curve gap became positive.

After the collapse of Lehman Brothers, in spite of lowering the policy rate almost to zero, monetary policy remained restrictive as measured by both, the short-term rate gap and the yield curve gap (1q2009 - panel c). Yet, the introducing of the QE programme led to a substantial lowering of the longer part of the yield curve making monetary conditions accommodative. According to the NYC model, monetary policy was most accommodative in 1q2010 (panel d). In contrast, the short end of the yield curve remained above its natural level in that period.

The zero lower bound problem and the implementation of non-standard monetary policy instruments like quantitative easing highlight some limitations for the short-term rate gap as a measure of the monetary policy stance. In our view the yield curve gap concept may fill this gap.

## 5 Conclusions

In this paper we define and estimate a new equilibrium concept in monetary economics - the natural yield curve (NYC). This can be seen as the whole term structure of natural rates of interest (NRI), which are a popular concept both in academia and at central banks. We motivate our innovation as follows. First, the NRI always provoked confusion as to what maturity it should be applied. While for several authors (including Knut Wicksell, its founder) it was a long-term concept, most recent studies approached the NRI from the short-term perspective. Our approach generalizes the NRI and allows for the calculation of short, medium and long-term natural rates.

Second, the usual argument for approaching the NRI from the short-term perspective was related to the fact that central banks control short-term interest rates. Hence, the respective NRI provides an important benchmark for their policies. However, the developments following the financial crisis 2007-09 changed the way central banks look at their policy instruments. After hitting the zero lower bound on interest rates several central banks extended their traditional operational policy framework and started to exert influence on the longer end of the yield curve. From this perspective the gap between the short-term real and natural rates of interest loses some appeal as a measure of the monetary policy stance. On the contrary, the gap between the real and natural yield curves becomes more appealing.

Using a framework similar to Laubach and Williams (2003) we show how to estimate the NYC on US data. We show that the gap between the real and natural curves has an impact on the output gap and inflation. Furthermore we demonstrate how, in contrast to the short-term interest rate gap, our gap is able to document the expansionary stance of the Fed's monetary policy during and after the financial crisis.

## References

- Amato, Jeffery D. (2005) ‘The role of the natural rate of interest in monetary policy.’ BIS Working Papers 171, Bank for International Settlements, March
- Andrés, Javier, David J. López-Salido, and Edward Nelson (2009) ‘Money and the natural rate of interest: Structural estimates for the united states and the euro area.’ *Journal of Economic Dynamics and Control* 33(3), 758–776
- Ang, Andrew, and Monika Piazzesi (2003) ‘A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variables.’ *Journal of Monetary Economics* 50(4), 745–787
- BIS (2005) *Zero-coupon yield curves: technical documentation* number 25. In ‘BIS Papers.’ (Bank for International Settlements)
- Blinder, Alan S., Michael Ehrmann, Marcel Fratzscher, Jakob De Haan, and David-Jan Jansen (2008) ‘Central bank communication and monetary policy: A survey of theory and evidence.’ *Journal of Economic Literature* 46(4), 910–45
- Bomfim, Antulio N. (2001) ‘Measuring equilibrium real interest rates: what can we learn from yields on indexed bonds?’ Finance and Economics Discussion Series 2001-53, Board of Governors of the Federal Reserve System (U.S.)
- Brzoza-Brzezina, Michał (2006) ‘The information content of the neutral rate of interest.’ *The Economics of Transition* 14(2), 391–412
- Christensen, Jens H. E., Francis X. Diebold, and Glenn D. Rudebusch (2009) ‘An arbitrage-free generalized nelson–siegel term structure model.’ *Econometrics Journal* 12(3), C33–C64
- Crespo Cuaresma, Jesús, Ernest Gnan, and Doris Ritzberger-Gruenwald (2004) ‘Searching for the natural rate of interest: a euro area perspective.’ *Economic Change and Restructuring* 31(2), 185–204
- Dai, Qiang, and Kenneth J. Singleton (2000) ‘Specification analysis of affine term structure models.’ *Journal of Finance* 55(5), 1943–1978
- Diebold, Francis X., and Canlin Li (2002) ‘Forecasting the term structure of government bond yields.’ Center for Financial Institutions Working Papers 02-34,

- Wharton School Center for Financial Institutions, University of Pennsylvania, August
- Diebold, Francis X., Glenn D. Rudebusch, and Aruoba S. Boragan (2006) ‘The macroeconomy and the yield curve: a dynamic latent factor approach.’ *Journal of Econometrics* 131(1-2), 309–338
- Duffee, Gregory R. (2002) ‘Term premia and interest rate forecasts in affine models.’ *Journal of Finance* 57(1), 405–443
- Duffie, Darrell, and Rui Kan (1996) ‘A yield-factor model of interest rates.’ *Mathematical Finance* 6(4), 379–406
- Edge, Rochelle M., Michael T. Kiley, and Jean-Philippe Laforte (2008) ‘Natural rate measures in an estimated dsge model of the u.s. economy.’ *Journal of Economic Dynamics and Control* 32(8), 2512–2535
- Garnier, Julien, and Bjorn-Roger Wilhelmsen (2005) ‘The natural real interest rate and the output gap in the euro area - a joint estimation.’ Working Paper Series 546, European Central Bank, November
- Gasha, Jose Giancarlo, Carlos I. Medeiros, Jean Salvati, Marco Rodriguez, Ying He, and Jiangbo Yi (2010) ‘On the estimation of term structure models and an application to the united states.’ IMF Working Papers 10/258, International Monetary Fund, November
- Gilli, Manfred, Stefan GroA Y”e, and Enrico Schumann (2010) ‘Calibrating the nelson-siegel-svensson model.’ Working Papers 031, COMISEF, March
- Haubrich, Joseph G., George Pennacchi, and Peter Ritchken (2008) ‘Estimating real and nominal term structures using treasury yields, inflation, inflation forecasts, and inflation swap rates.’ Technical Report
- Laubach, Thomas, and John C. Williams (2003) ‘Measuring the natural rate of interest.’ *The Review of Economics and Statistics* 85(4), 1063–1070
- Mesonnier, Jean-Stephane, and Jean-Paul Renne (2007) ‘A time-varying “natural” rate of interest for the euro area.’ *European Economic Review* 51(7), 1768–1784
- Nelson, Charles R, and Andrew F Siegel (1987) ‘Parsimonious modeling of yield curves.’ *Journal of Business* 60(4), 473–89

- Orphanides, Athanasios, and John C. Williams (2007) ‘Robust monetary policy with imperfect knowledge.’ *Journal of Monetary Economics* 54(5), 1406–1435
- Pooter, Michiel De (2007) ‘Examining the nelson-siegel class of term structure models.’ Tinbergen Institute Discussion Papers 07-043/4, Tinbergen Institute, June
- Rudebusch, GlennD., and Tao Wu (2008) ‘A macro-finance model of the term structure, monetary policy and the economy.’ *Economic Journal* 118(530), 906–926
- Stock, James H., and Mark W. Watson (1998) ‘Median unbiased estimation of coefficient variance in a time-varying parameter model.’ *Journal of the American Statistical Association* 93(441), pp. 349–358
- Svensson, Lars E.O. (1994) ‘Estimating and interpreting forward interest rates: Sweden 1992 - 1994.’ Working Paper 4871, National Bureau of Economic Research, September
- Wicksell, Knut (1898) *Interest and Prices* (Translated by R.F. Kahn, 1936. MacMillan)
- Woodford, Michael (2003) *Interest and Prices : Foundations of a Theory of Monetary Policy* (Princeton University Press)

## Tables and Figures

Table 1: Estimation results for DNS model

State Equations			
	$L_t$	$S_t$	$C_t$
$\mu$	0.0366 (0.0085)	-0.0354 (0.0060)	-0.0276 (0.0040)
$L_{t-1}$	0.9885 (0.0270)	-0.0582 (0.0433)	-0.1591 (0.0696)
$S_{t-1}$	0.0851 (0.0259)	0.7228 (0.0441)	-0.0755 (0.0755)
$C_{t-1}$	-0.0661 (0.0215)	0.2568 (0.0406)	0.9232 (0.0593)
Residual Covariance Matrix			
	$L_t$	$S_t$	$C_t$
$L_t$	$8.53 \cdot 10^{-6}$ ( $9.57 \cdot 10^{-8}$ )	$8.46 \cdot 10^{-6}$ ( $1.47 \cdot 10^{-6}$ )	$5.97 \cdot 10^{-6}$ ( $2.30 \cdot 10^{-6}$ )
$S_t$		$2.47 \cdot 10^{-5}$ ( $2.71 \cdot 10^{-6}$ )	$3.16 \cdot 10^{-7}$ ( $3.61 \cdot 10^{-6}$ )
$C_t$			$6.00 \cdot 10^{-5}$ ( $8.96 \cdot 10^{-7}$ )

Table 2: Estimation results for the NYC model - measurement equations

	IS Curve	Phillips Curve
	$Output\_Gap_t$	$Core\_CPI_t$
$Output\_Gap_{t-1}$	0.644 (0.014)	0.104 (0.001)
$Output\_Gap_{t-2}$	0.286 (0.003)	-
$Curve\_Gap_{t-1}$	-0.0498 (0.0003)	-
$Core\_CPI_{t-1}$	-	0.880 (0.087)
$Core\_CPI_{t-2}$	-	0.120
<i>Residual Variance</i>	$2.61 \cdot 10^{-5}$	$5.63 \cdot 10^{-6}$

Table 3: Estimation results for the NYC model - state equations

	Natural Slope Equation	Potential Output Equation
	$Natural\_Slope_t$	$Potential\_Output_t$
$Natural\_Slope_{t-1}$	0.932 (0.004)	-
$Potential\_Output_t$	-	1.271 (0.022)
$Potential\_Output_t$	-	-0.271
$Const$	-	0.005 (0.0003)
<i>Residual Variance</i>	$3.72 \cdot 10^{-5}$	$7.50 \cdot 10^{-6}$

Figure 1: Loadings in the Nelson-Siegel model

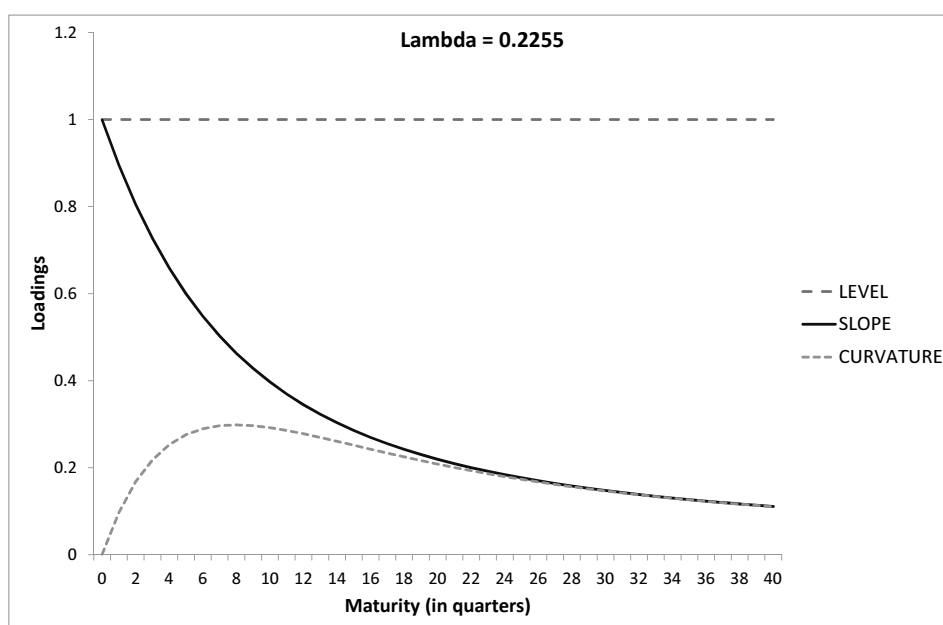
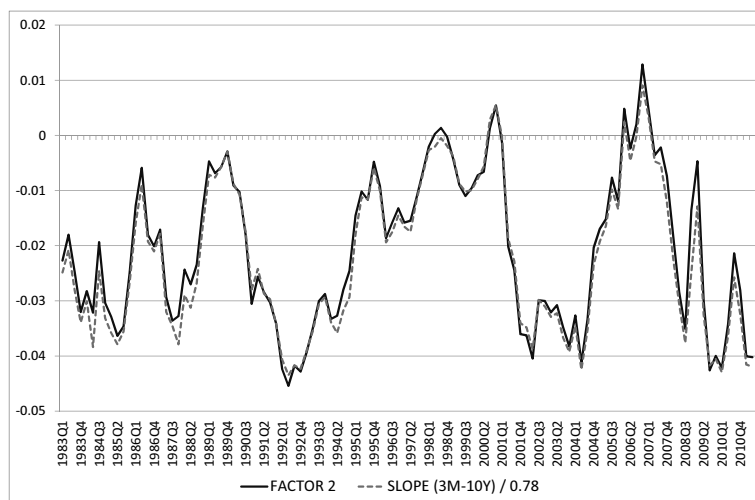


Figure 2: The level factor and its empirical counterpart



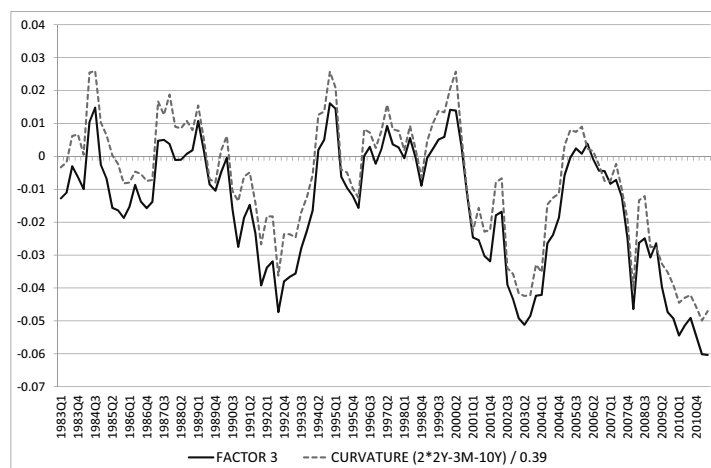
Note: the empirical counterpart is the real 10-year yield. For explanation see footnote 7.

Figure 3: The slope factor and its empirical counterpart



Note: the empirical counterpart is the spread between the 3-month and 10-year yield.  
For explanation see footnote 7.

Figure 4: The curvature factor and its empirical counterpart



Note: the empirical counterpart is the difference between twice the 2-year yield and the sum of the 3-month and 10-year yields. For explanation see footnote 7.

Figure 5: The natural slope

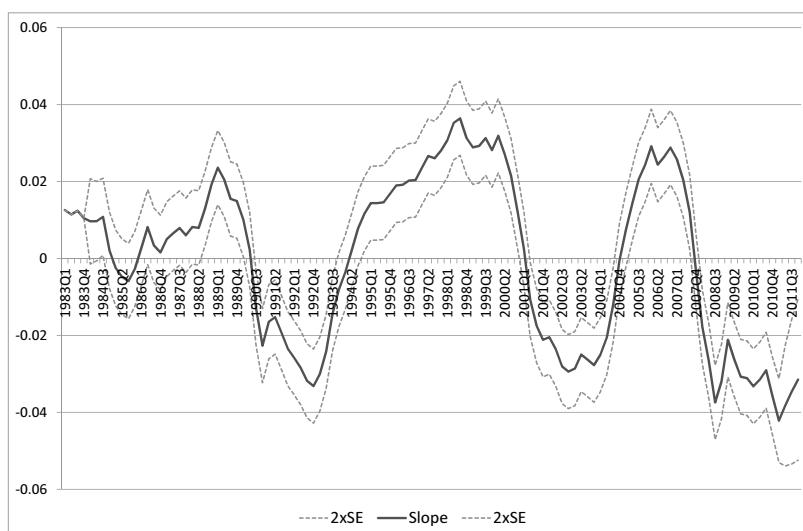


Figure 6: Potential output

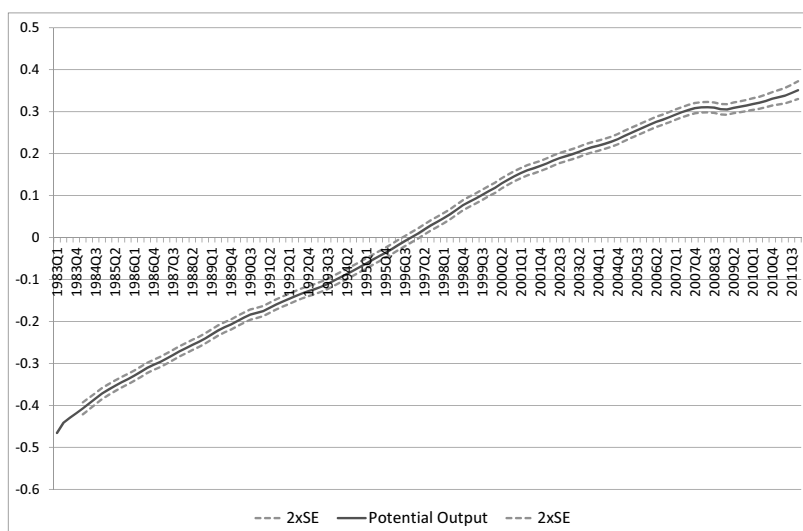


Figure 7: Estimates of the output gap from the NYC model and CBO's output gap

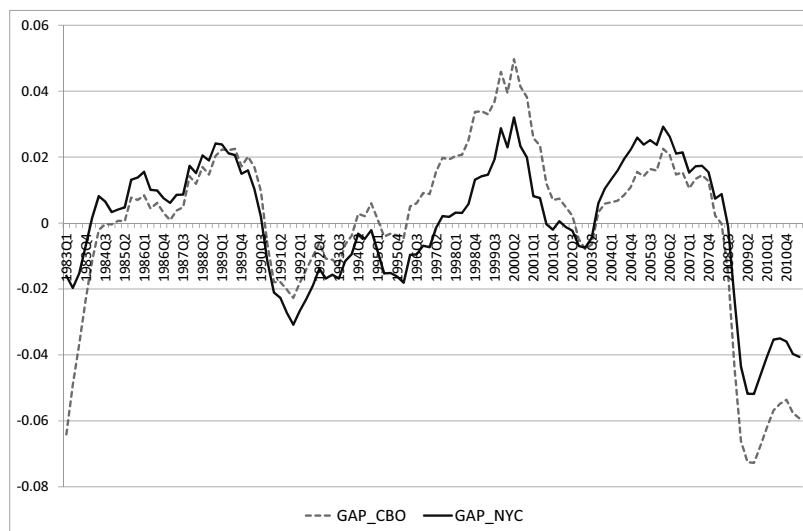


Figure 8: Yield curve gap vs. output gap

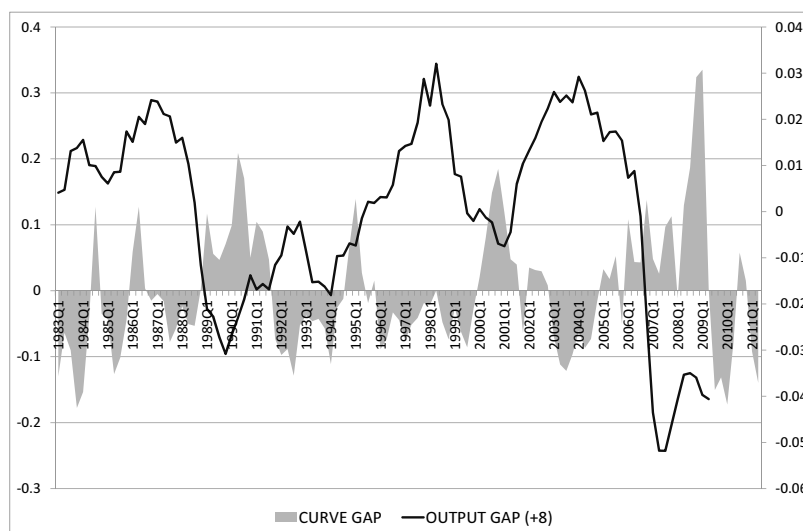


Figure 9: Yield curve gap vs. core inflation

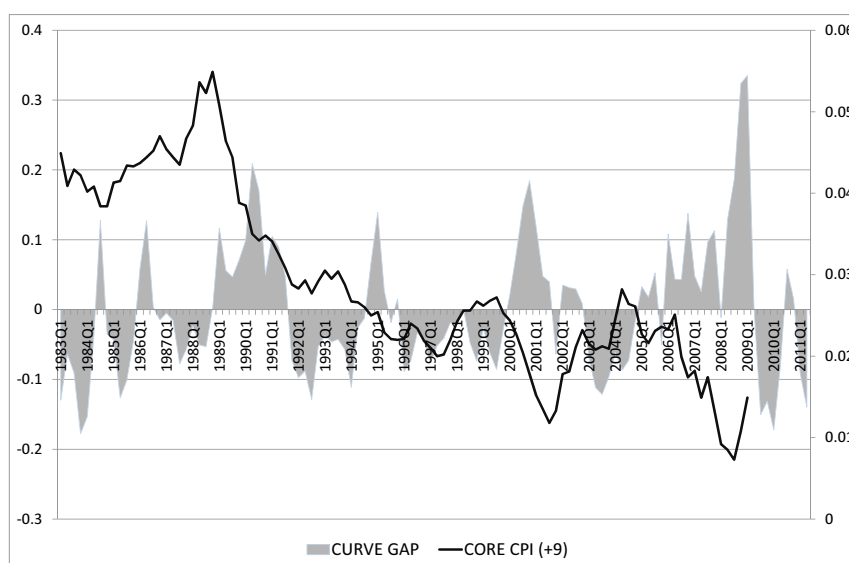


Figure 10: Yield curve gap vs. short term rate gap

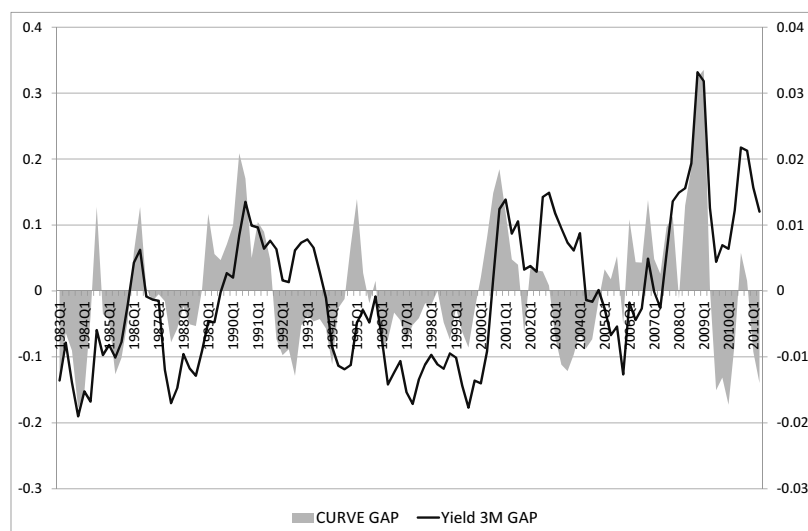


Figure 11: The natural yield curve

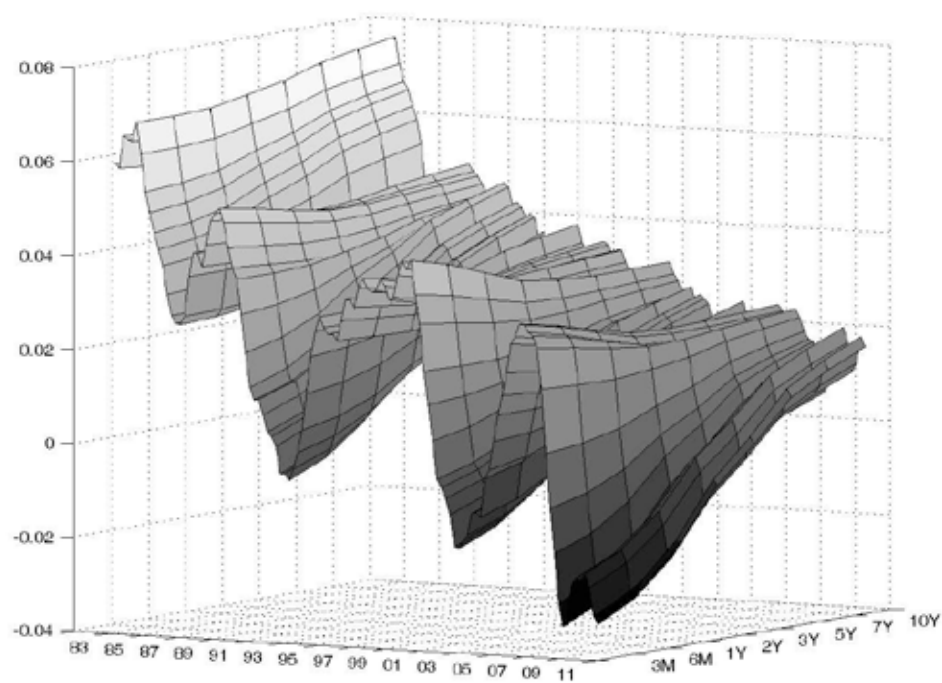


Figure 12: Real and natural yield curves

