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On the empirical importance of periodicity
in the volatility of financial time series

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Abstract

We discuss the empirical importance of long term cyclical effects in the volatility of financial returns. Following Čížek and Spokoiny (2009), Amado and Teräsvirta (2012) and others, we consider a general conditionally heteroscedastic process with stationarity property distorted by a deterministic function that governs the possible variability in time of unconditional variance. The function proposed in this paper can be interpreted as a finite Fourier approximation of an Almost Periodic (AP) function as defined by Corduneanu (1989). The resulting model has a particular form of a GARCH process with time varying parameters, intensively discussed in the recent literature.

In the empirical analyses we apply a generalisation of the Bayesian AR(1)- t -GARCH(1,1) model for daily returns of S&P500, covering the period of sixty years of US postwar economy, including the recently observed global financial crisis. The results of a formal Bayesian model comparison clearly indicate the existence of significant long term cyclical patterns in volatility with a strongly supported periodic component corresponding to a 14 year cycle. This may be interpreted as empirical evidence in favour of a linkage between the business cycle in the US economy and long term changes in the volatility of the basic stock market index.

Keywords: Periodically correlated stochastic processes, GARCH models, Bayesian inference, volatility, unconditional variance

JEL classification: C58, C11, G10

1 Introduction

Starting from seminal works by Clark (1973), Engle (1982) and Bollerslev (1986) stochastic processes used to describe observed properties of the volatility of financial time series have been tailored to identify short term features. In particular, the resurgence of stochastic volatility (SV) models in the 90's relied on the assumption that there exists a stochastic factor independent of the past of the process, which influences volatility in the short term. The resulting literature concerning GARCH and SV models, its properties and practical importance is enormous, however empirical analyses of the dynamic behaviour of the volatility in the long term has not been fully explored so far.

Recently, some attempts to model long term features of volatility have been made. Since empirical analyses of long time series of financial returns clearly indicated that parameters of volatility models may vary over time, it is obvious that models applied so far may not capture properties of volatility which are important in the long term. At the beginning of the 90's the GARCH-type models became a very popular tool of volatility modelling. But parallelly some problems were identified with their applications to long time series of financial returns. For example, Lamoreux and Lastrapes (1990) and Engle and Mustafa (1992) suggested that parameters of GARCH-type processes are very strongly identified, because while in econometric applications their estimates are statistically highly significant, they are not stable over time. Consequently, the constancy of parameters initially imposed in GARCH-type processes was subject to criticism that prompted new studies concerning generalisations. In particular Mikosh and Stărică (2004) indicate that the IGARCH effect is often spuriously supported by data, because in the case of long time series variability of parameters is natural. Hence the regular GARCH(1,1) structure is unable to capture nonlinearity and possible complex stochastic properties of the observed process. Teräsvirta (2009) points out a more formal motivation in favour of time variability of parameters in a parametric GARCH scheme, suggesting that *constancy of parameters can be a testable restriction and if it is rejected, the model should be generalised*. Several approaches

have been proposed imposing time variability of parameters in volatility models. We see two basic fundamental approaches applied in this respect, the first one relates to variability governed by a random process, and the second relies on deterministic framework. Within the confines of the first approach, Hamilton and Susmel (1994) conducted research on the empirical importance of the assumption that stock returns are characterised by different ARCH processes at different points in time, with the shifts between processes mediated by a Markov chain. This straightforward approach opened new topics in financial econometrics based on the application of Markov switching mechanisms in volatility modelling. A possible variability of parameters described by a deterministic function was also subject to analysis. Teräsvirta (2009) modified the smooth transition GARCH model by imposing a transition function of the form that guarantees variability of parameters for a process observed in finite time interval. The transition function depends on the length of the observed time series. Čížek and Spokoiny (2009) present a review of literature concluding that relaxing time homogeneity of the process is a promising approach but causes serious problems with proper estimation methods. For instance, when some or all model parameters will vary over time, a more subtle treatment of testing structural breaks in financial returns may be obtained; see Fan and Zhang (1999), Cai et al. (2000), Fan et al. (2003). An approach to the specification of time varying GARCH models was developed in the field of nonparametric statistics. Under very general conditions concerning the regularity of parameters treated as functions of time, nonparametric methods of estimation were proposed; see Härdle et al. (2003), Mercurio and Spokoiny (2004), Spokoiny and Chen (2007) and Čížek and Spokoiny (2009).

The main purpose of this paper is to propose a simple generalisation of the GARCH model which would enable to model long term features of volatility. Our construct is strictly related to the literature studying the properties of GARCH processes with time varying parameters and is based on parametric approach; see Teräsvirta (2009), Amado and Teräsvirta (2012). The variability of unconditional moments is governed by a class of Almost Periodic (AP) functions, proposed by Corduneanu (1989) as a generalisation of the class of periodic functions. Since in our approach the unconditional second

moment exhibits almost periodic variability, the process can be also interpreted as a second order Almost Periodically Correlated (APC) stochastic process, discussed from the theoretical point of view by Hurd and Miamee (2007). During the last five decades the APC class of processes was broadly applied in telecommunication (Gardner (1986), Napolitano and Spooner (2001)), climatology (Bloomfield et al. (1994)) and many other fields. For an exhaustive review of possible applications see Gardner et al. (2006).

We make a formal statistical inference, from the Bayesian viewpoint, about the cyclicity of volatility changes and present evidence in favour of the empirical importance of such an effect. On the basis of very intuitive explanation of almost periodicity, we provide an economic interpretation of time variability of unconditional moments supported by data. The illustration is conducted on the basis of daily returns of the S&P500 index covering the period from 18 January 1950 till 7 February 2012.

2 A simple nonstationary process obtained from the GARCH(1,1) model

We start from a general definition of a conditionally heteroscedastic model which nests many GARCH-type volatility models developed during more than the last four decades in the field of financial econometrics.

Definition 1. Discrete, real valued, stochastic process $\{\xi_t, t \in Z\}$ is called conditionally heteroscedastic if:

$$\xi_t = \sqrt{h_t(\omega, \Psi_{t-1})} z_t, \quad z_t \sim iid,$$

where $h_t(\omega, \Psi_{t-1})$ constitutes volatility equation, defined as a parametric function of the information set $\Psi_{t-1} = (\dots, \xi_{t-2}, \xi_{t-1})$, i.e. the history of the process $\{\xi_t, t \in Z\}$, and parameters ω . Random variables z_t are identically and independently distributed.

Any conditionally heteroskedastic GARCH-type model, defined in the literature, starting from the ARCH(p) model proposed by Engle (1982) and the GARCH(p, q), proposed by Bollerslev (1986), can be obtained by imposing some particular functional

form of $h_t(\omega, \Psi_{t-1})$.

For further analysis let us consider the discrete and real valued stochastic process $\{\varepsilon_t, t \in Z\}$ defined as follows:

$$\varepsilon_t = \sqrt{g(t, \gamma)} \xi_t, \quad (1)$$

where $\{\xi_t, t \in Z\}$ is defined by Definition 1, and $g(., \gamma)$ is a positive real valued function of time domain Z , parameterised by γ . The following theorem presents obvious necessary and sufficient conditions of moment existence for the process $\{\varepsilon_t, t \in Z\}$. The form of the process $\{\varepsilon_t, t \in Z\}$ is related to the general specification considered by Amado and Teräsvirta (2012). The aim of our study is a proper specification of function g , so that it has an economic interpretation and is empirically important.

Theorem 1. For a process $\{\varepsilon_t, t \in Z\}$ in (1), where $\{\xi_t, t \in Z\}$ is given by Definition 1 we have the following equivalences:

1. For each $n \in N$, $E(\varepsilon_t^n)$ exists and $E(\varepsilon_t^n) = g(t, \gamma)^{\frac{n}{2}} E(\xi_t^n)$ if and only if $E(\xi_t^n)$ exists
2. For each $n \in N$, $E(\varepsilon_t^n | \Psi_{t-1})$ exists and $E(\varepsilon_t^n | \Psi_{t-1}) = g(t, \gamma)^{\frac{n}{2}} h_t(\theta, \Psi_{t-1})^{\frac{n}{2}} E(z_t^n)$ if and only if $E(z_t^n)$ exists.

As an example of the process in Definition 1 let us consider the seminal Generalised Autoregressive Conditional Heteroskedastic (GARCH) process, initially defined by Bollerslev (1986). Formally Bollerslev (1986) defined GARCH(p, q) process for any natural p and q by defining lag of squared residuals and conditional variance applied in the volatility equation. According to Engle (1985) who stated that *the GARCH(1,1) is the leading generic model for almost all asset classes of returns [...] it is quite robust and does most of the work in almost all cases*, we focus our attention on the case with $p = 1$ and $q = 1$. In the definition of GARCH(1,1) process we follow Bauwens, Lubrano and Richard (1999) setting.

Definition 2. A discrete, real valued, stochastic process $\{\xi_t, t \in Z\}$ is called GARCH(1,1) if:

$$\xi_t = \sqrt{h_t} z_t,$$

where

$$h_t = \alpha_0 + \alpha_1 \xi_{t-1}^2 + \beta_1 h_{t-1},$$

for $\alpha_0 \geq 0$, $\alpha_1 > 0$, $\beta_1 > 0$. Random variables z_t are identically and independently distributed in such a way, that:

1. $E(z_t) = 0$
2. $E(z_t^2) = 1$
3. $E(z_t^3) = 0$
4. $E(z_t^4) = \lambda > 0$.

When analysing stochastic properties of the process in Definition 2, it is crucial to pay attention on the restriction $\alpha_1 + \beta_1 < 1$. It ensures moment existence up to the second order and its stability over time and, consequently, covariance stationarity of the process. The GARCH(1,1) process with $\alpha_1 + \beta_1 = 1$ is called IGARCH. This case still represents the process stationary in the strict sense, but covariance stationarity is no longer fulfilled. Bauwens, Lubrano and Richard (1999) listed the following properties of the GARCH(1,1) process:

Theorem 2. If the process $\{\xi_t, t \in Z\}$ follows Definition 2, then:

1. $E(\xi_t | \Psi_{t-1}) = 0$
2. $V(\xi_t) = E(\xi_t^2 | \Psi_{t-1}) = h_t$
3. $E(\xi_t^3 | \Psi_{t-1}) = 0$
4. $E(\xi_t^4 | \Psi_{t-1}) = \lambda h_t^2$
5. $K(\xi_t^4 | \Psi_{t-1}) = \lambda$
6. $E(\xi_t) = 0$
7. $E(\xi_t^2) = \frac{\alpha_0}{1 - \alpha_1 - \beta_1}$ if additionally $\alpha_1 + \beta_1 < 1$
8. $E(\xi_t^3) = 0$

9. $E(\xi_t^4) = \lambda \alpha_0^2 \frac{1 + \alpha_1 + \beta_1}{(1 - \lambda \alpha_1^2 - \beta_1^2 - 2\alpha_1\beta_1)(1 - \alpha_1 - \beta_1)}$ if additionally $\alpha_1 + \beta_1 < 1$ and $\lambda \alpha_1^2 + \beta_1^2 + 2\alpha_1\lambda < 1$
10. $K(\xi_t) = \lambda \frac{(1 - \alpha_1 - \beta_1)(1 + \alpha_1 + \beta_1)}{1 - \lambda \alpha_1^2 - \beta_1^2 - 2\alpha_1\beta_1}$
11. $Corr(\xi_t^2, \xi_{t-1}^2) = \frac{\alpha_1(1 - \beta_1^2 - \alpha_1\beta_1)}{1 - \beta_1^2 - 2\alpha_1\beta_1}$ if additionally $\alpha_1 + \beta_1 < 1$
12. $Corr(\xi_t^2, \xi_{t-k}^2) = (\alpha_1 + \beta_1)Corr(\xi_t^2, \xi_{t-k-1}^2) = (\alpha_1 + \beta_1)^{k-1} \frac{\alpha_1(1 - \beta_1^2 - \alpha_1\beta_1)}{1 - \beta_1^2 - 2\alpha_1\beta_1}$ if additionally $\alpha_1 + \beta_1 < 1$,

where $K(\xi)$ denotes kurtosis for a random variable ξ and $Corr(\xi_1, \xi_2)$ denotes correlation between ξ_1 and ξ_2 .

According to Theorem 2, given restriction $\alpha_1 + \beta_1 < 1$, process $\{\xi_t, t \in Z\}$ is covariance stationary with unconditional zero mean and finite unconditional variance $V(\xi_t) = E(\xi_t^2) = \frac{\alpha_0}{1 - \alpha_1 - \beta_1}$. Also, autocovariance function for any nonzero lag is equal to zero. However for squares of the process, again if $\alpha_1 + \beta_1 < 1$, autocorrelation function is not constant and decays at exponential rate as a function of lag. Now let consider the process $\{\varepsilon_t, t \in Z\}$ defined by (1), generated by the GARCH(1,1) process $\{\xi_t, t \in Z\}$. Automatically from Theorem 1 we obtain the following properties:

Theorem 3. If the process $\{\varepsilon_t, t \in Z\}$ is defined by equation (1) and $\{\xi_t, t \in Z\}$ in (1) follows Definition 2, then:

$$E(\varepsilon_t | \Psi_{t-1}) = 0$$

$$E(\varepsilon_t^2 | \Psi_{t-1}) = g(t, \gamma) h_t$$

$$E(\varepsilon_t^3 | \Psi_{t-1}) = 0$$

$$E(\varepsilon_t^4 | \Psi_{t-1}) = \lambda g(t, \gamma)^2 h_t^2$$

$$K(\varepsilon_t | \Psi_{t-1}) = \lambda$$

$$E(\varepsilon_t) = 0$$

$$V(\varepsilon_t) = E(\varepsilon_t^2) = g(t, \gamma) \frac{\alpha_0}{1 - \alpha_1 - \beta_1}, \text{ if additionally } \alpha_1 + \beta_1 < 1$$

$$E(\varepsilon_t^3) = 0$$

$$E(\varepsilon_t^4) = g(t, \gamma)^2 \lambda \alpha_0^2 \frac{1 + \alpha_1 + \beta_1}{(1 - \lambda \alpha_1^2 - \beta_1^2 - 2\alpha_1 \lambda)(1 - \alpha_1 - \beta_1)} \text{ if additionally } \alpha_1 + \beta_1 < 1$$

and $\lambda \alpha_1^2 + \beta_1^2 + 2\alpha_1 \lambda < 1$

$$K(\varepsilon_t) = \lambda \frac{(1 - \alpha_1 - \beta_1)(1 + \alpha_1 + \beta_1)}{1 - \lambda \alpha_1^2 - \beta_1^2 - 2\alpha_1 \lambda} = K(\xi_t)$$

$$\text{Corr}(\varepsilon_t^2, \varepsilon_{t-1}^2) = \text{Corr}(\xi_t^2, \xi_{t-1}^2) \text{ if additionally } \alpha_1 + \beta_1 < 1$$

$$\text{Corr}(\varepsilon_t^2, \varepsilon_{t-k}^2) = \text{Corr}(\xi_t^2, \xi_{t-k}^2) \text{ if additionally } \alpha_1 + \beta_1 < 1.$$

It is clear from Theorem 3, that process $\{\varepsilon_t, t \in Z\}$ is nonstationary in the strict sense and also covariance nonstationary. Function $g(f(\cdot, \gamma))$ assures variability of unconditional moments, especially the unconditional variance given by formula 7 in Theorem 3. Also variability over time of conditional variance of y_t is decomposed into GARCH(1,1) effect and deterministic component, that changes dispersion of the conditional distribution according to the form of function g . From the definition of process $\{z_t, t \in Z\}$ we keep the first and the third moment, and also kurtosis of ε_t , unchanged. Consequently our construct generates a nonstationary process with time-varying unconditional second order moments and with an additional source of variability of the conditional variance.

Another interesting feature of $\{\varepsilon_t, t \in Z\}$ can be observed if we rewrite the equation for conditional variance in GARCH-type form. If the process $\{\varepsilon_t, t \in Z\}$ is defined by equation (1) and $\{\xi_t, t \in Z\}$ in (1) follows Definition 2, we have:

$$E(\varepsilon_t^2 | \Psi_{t-1}) = g(t, \gamma) h_t = \alpha_{0,t} + \alpha_{1,t} \xi_{t-1}^2 + \beta_{1,t} h_{t-1}, \quad (2)$$

where $\alpha_{0,t} = g(t, \gamma) \alpha_0$, $\alpha_{1,t} = g(t, \gamma) \alpha_1$ and $\beta_{1,t} = g(t, \gamma) \beta_1$. Hence, the process $\{\varepsilon_t, t \in Z\}$ can be also interpreted as a GARCH(1,1) model with time varying parameters.

3 A model for periodic volatility

The main purpose of the paper is a proper definition of function g in (1), that would enable the testing the variability of parameters in (2) but also provide an economic interpretation of such an effect. The vast literature concerning time-varying GARCH models does not seem to explore this aspect in detail, focusing only on the statistical

properties of estimation methods, given very general assumptions about the variability of parameters.

Some attempts to interpret time heterogeneity of processes describing volatility have been made. One of them was adopted by Hamilton (1989). In this seminal paper formal statistical representation of the old idea that expansion and contraction constitute two distinct economic phases was considered. Hamilton proposed to model real output growth by two autoregressions, depending on whether the economy is expanding or contracting. Possible changes between those autoregressions were governed by a Markov chain. The main contribution of Hamilton (1989) was a very intuitive economic interpretation of a purely random construct as a factor governing changes between states of different intensity of economic activity. This idea was easily instilled in modelling financial time series, where Markov switching ARCH and GARCH models were specifically developed for volatility modelling; see Hamilton and Susmel (1994), Susmel (2000), Haas et al. (2004), Li and Lin (2004) and many others. Just like in the case of the business cycle, Markov switching volatility models are able to distinguish phases of low and high volatility, or - in the case of many regimes - many different levels of risk intensity. However, as concluded by Langa and Rahbek (2009), in spite of the fact that Markov switching volatility models have recently received much interest in applications, a sufficiently complete theory for these models is still missing. Some theoretical aspects concerning properties of forecasts were presented; see for example Amendola and Niglio (2004). Yet, in Markov switching conditionally heteroscedastic models basic conditions that would guarantee stationarity are still unknown.

Another disadvantage of Hamilton's approach is that different phases of intensity of volatility change abruptly, according to a random process in time domain. The resulting path of the volatility process has discontinuities at points where a change of regime was observed. This leaves the application of the whole Hamilton's idea in volatility modelling with doubts. Analogously to modelling economic activity of the real sector, the volatility of financial time series also seems to have phases of expansion and contraction in the long term. An analysis of financial returns in the span of decades shows that changes between states are much closer to continuous rather than

discrete. Since those phases alternate in cases of boom and bust on the market, on a large scale volatility should also exhibit cyclical behaviour. In order to test for such an effect, a stochastic process with an approximately periodic structure of unconditional moments should be considered. For a process $\{\varepsilon_t, t \in Z\}$ defined by equation (1), where $\{\xi_t, t \in Z\}$ in (1) follows Definition 2, it can easily be done on the basis of an appropriately defined function $g(\cdot, \gamma)$, which describes the variability of moments. In general, we follow the idea of generalisation of periodicity of real valued functions proposed by Corduneanu (1989).

Definition 3. A real-valued function $f : Z \rightarrow \mathbb{R}$ of an integer variable is called almost periodic (AP in short), if for any $\epsilon > 0$ there exists an integer $L_\epsilon > 0$, such that among any L_ϵ consecutive integers, there is an integer p_ϵ with the property

$$\sup_{t \in \mathbb{Z}} |f(t + p_\epsilon) - f(t)| < \epsilon.$$

Any periodic function is also almost periodic. Conditions from Definition 3 constitute a class of almost periodically correlated (APC) stochastic processes as a generalisation of periodically correlated (PC) stochastic processes. In the case of APC processes, an almost periodic function, and in the case of PC processes, a periodic function, determines the cyclical variability of conditional and unconditional moments. Therefore PC stochastic processes are also called cyclostationary. During the last five decades the APC class was broadly applied in telecommunication (Gardner (1986)), Napolitano and Spooner (2001), climatology (Bloomfield et al. (1994)) and many other fields. For exhaustive review of possible applications see Gardner et al. (2006).

The main properties of the APC class was presented by Corduneanu (1989). In particular, any almost periodic function from Definition 3 has its unique Fourier expansion of the form:

$$f(t) = \sum_{i=1}^{\infty} (g_{si} \sin(h_i t) + g_{ci} \cos(h_i t)), \quad (3)$$

with the series of coefficients $(g_{si})_{i=1}^{\infty}$, $(g_{ci})_{i=1}^{\infty}$ and $(h_i)_{i=1}^{\infty}$ that express amplitude and

frequency of individual cyclical component in (3).

For further research concerning cyclical behavior of volatility, we consider the following function $g(., \gamma)$ in (1):

$$g(t, \gamma) = e^{f(t, \gamma)}, \quad (4)$$

where

$$f(t, \gamma) = \sum_{i=1}^F (\gamma_{si} \sin(\phi_i t) + \gamma_{ci} \cos(\phi_i t)), \quad (5)$$

with $\gamma = (\gamma_{s1}, \dots, \gamma_{sF}, \gamma_{c1}, \dots, \gamma_{cF}, \phi_1, \dots, \phi_F)$. Function $f(., \gamma)$ is defined as a sum of periodic functions, with parameters ϕ_i determining frequencies, while γ_{si} and γ_{ci} control amplitudes. Since we limit the infinite series to its finite substitute, formula (5) yields finite approximation of order F of the almost periodic function, that governs moment variability of the process.

The case $\gamma_{si} = 0$ and $\gamma_{ci} = 0$ for all $i = 1, \dots, F$, in (5), determines constant function $g(., \gamma) \equiv 1$. According to Hurd and Miamee (2007), the process $\{\varepsilon_t, t \in Z\}$ defined by equation (1), where $\{\xi_t, t \in Z\}$ in (1) follows Definition 2 is also Almost Periodically Correlated. According to the properties shown in the previous section, the function $g(., \gamma)$ in (5) enables to model the existence of cyclicity in the conditional and unconditional variance of the process. This property will be subject to formal statistical inference in the empirical part of the paper.

4 Basic model framework and posterior inference

We model logarithmic returns on the financial instrument with price x_t at time t . Suppose, we observe time series of logarithmic returns given by the form:

$$y_t = 100 \ln \frac{x_t}{x_{t-1}}, t = -1, 0, 1, \dots, T.$$

Denote by $y = (y_1, \dots, y_T)$ the vector of modelled observations. Daily returns y_{-1} and y_0 are used as initial values.

Just like many authors, in order to model the dynamics of financial returns we assume an AR(1) process with nonstationary disturbances of the following form; see for example

Bauwens, Lubrano and Richard (1999) for univariate case or Osiewalski and Pipień (2004) for multivariate setting:

$$y_t = \mu_t + \varepsilon_t, \quad t = 1, \dots, T, \quad (6)$$

where $\mu_t = \delta + \rho(y_{t-1} - \delta)$ and ε_t is a process defined by (3), i.e.:

$$\varepsilon_t = \sqrt{f(t, \gamma)} \xi_t, \quad t = 1, \dots, T,$$

with the GARCH(1,1) process ξ_t from Definition 2:

$$\xi_t = \sqrt{h_t} z_t, \quad h_t = \alpha_0 + \alpha_1 \xi_{t-1}^2 + \beta_1 h_{t-1} \quad t = 1, \dots, T,$$

and function f given by (3). We assume, that random variables z_t in Definition 2 are independent and follow Student- t distribution with zero mean, unit variance and $\nu > 4$ degrees of freedom. In the literature, conditional Student- t distribution is applied in GARCH models, with standard restriction $\nu > 2$ imposed, that assures existence of conditional variance. However, in order to keep all conditions in definition 2 fulfilled, we have to assume, that $\nu > 4$. Given this stronger restriction $E(z_t^4) = \lambda = 3 \frac{\nu - 2}{\nu - 4}$. The density of z_t is given by the formula:

$$f_s(z_t|0, 1, \nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi(\nu-2)}} \left[1 + \frac{z_t^2}{\nu-2} \right]^{-\frac{\nu+1}{2}},$$

where $f_s(z_t|m, s^2, \nu)$ denotes the density of the Student- t distribution with mean m , variance s^2 and $\nu > 4$ degrees of freedom.

The conditional distributions of ξ_t and ε_t are Student- t distributions with zero mean, $\nu > 4$ degrees of freedom and variances h_t and $g(t, \gamma)h_t$ respectively:

$$p(\xi_t|\Psi_{t-1}) = f_s(\xi_t|0, h_t, \nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi(\nu-2)h_t}} \left[1 + \frac{\xi_t^2}{(\nu-2)h_t} \right]^{-\frac{\nu+1}{2}},$$

$$p(\varepsilon_t|\Psi_{t-1}) = f_s(\varepsilon_t|0, g(t, \gamma)h_t, \nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi(\nu-2)g(t, \gamma)h_t}} \left[1 + \frac{\varepsilon_t^2}{(\nu-2)g(t, \gamma)h_t} \right]^{-\frac{\nu+1}{2}},$$

where $\Psi_0 = (h_0, y_{-1}, y_0)$, and $\Psi_{t-1} = (\Psi_0, y_1, \dots, y_{t-1}) = (\Psi_0, y^{(t-1)})$. Consequently, the conditional distribution of daily return in (6) is Student- t distribution with mean $\mu_t = \delta + \rho(y_{t-1} - \delta)$, variance $g(t, \gamma)h_t$ and $\nu > 4$ degrees of freedom:

$$p(y_t | \Psi_{t-1}) = f_s(y_t | \mu_t, g(t, \gamma)h_t, \nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi(\nu-2)g(t, \gamma)h_t}} \left[1 + \frac{(y_t - \mu_t)^2}{(\nu-2)g(t, \gamma)h_t} \right]^{-\frac{\nu+1}{2}}. \quad (7)$$

Let θ denote the vector, that contains all model parameters. We assume, that $\theta = (\mu', \sigma^2', \nu, \gamma')$ where vectors $\mu = (\delta, \rho)'$, $\sigma^2 = (\alpha_0, \alpha_1, \beta_1)'$ and $\gamma = (\gamma_{s1}, \dots, \gamma_{sF}, \gamma_{c1}, \dots, \gamma_{cF}, \phi_1, \dots, \phi_F)$ collect parameters of the conditional mean of y_t , the conditional variance of ε and function g respectively. According to (7), we define the sampling model for a vector y as follows:

$$p(y|\theta) = \prod_{t=1}^T p(y_t | \Psi_{t-1}) = \prod_{t=1}^T f_s(y_t | \mu_t, g(t, \gamma)h_t, \nu).$$

The Bayesian model, i.e. the joint distribution of observables and parameters, requires specification of the prior distribution $p(\theta)$. We assume the following prior independence:

$$p(y, \theta) = p(y|\theta)p(\theta) = p(y|\theta)p(\delta)p(\rho)p(\alpha_0)p(\alpha_1, \beta_1)p(\nu)p(\gamma),$$

where $p(\delta)$ is normal distribution, $p(\rho)$ is uniform over $(-1, 1)$, $p(\alpha_0)$ is exponential distribution with unit mean, $p(\alpha_1, \beta_1)$ is bivariate uniform distribution on the unit square $[0, 1]^2$, $p(\nu)$ is exponential distribution with mean 10 truncated at $\nu > 4$, $p(\gamma)$ is multivariate normal, $p(\phi)$ is multivariate uniform over the set generated by identification restrictions that eliminates label-switching effect in (5). We assume, that $L < \phi_1 < \dots < \phi_F < U$, for appropriately chosen L and U , which eliminates frequencies of length shorter than a quarter and longer than the time interval covering the observed time series.

5 Empirical analysis

In this section we present the empirical analysis and make formal Bayesian inference about the empirical importance of the cyclical component in the volatility of daily returns of one of the most important US Stock Market indices. Our dataset consists of $T = 15615$ observations of daily logarithmic returns of the S&P500 index, covering the period starting from the postwar era of the US economy till the beginning of 2012. The time series starts on 18 January 1950 and ends on 7 February 2012.

In Table 1 we present results of Bayesian model comparison, conducted for four competing specifications. Initially we consider conditionally a Student- t AR(1)-GARCH(1,1) process (denoted by M_0). We also add APC(F)-GARCH(1,1) models (denoted by M_F) for $F=1,2$ and 3 frequencies describing the variability of unconditional moments according to (5). In Table 1 we show decimal logarithms of marginal data densities, approximated by the Newton and Raftery (1994) estimator and decimal logarithms of the Bayes factor in favour of the best model. The dataset strongly supports time variability of parameters in GARCH(1,1) specification and consequently nonstationarity in the strict sense. The original GARCH(1,1) model receives little data support, as the marginal data density value is more than three orders of magnitude lower than in the case of the worst APC model, i.e. in the case when unconditional variance is described by a single cyclical component. The strongest data support goes to APC(2)-GARCH(1,1) model, with the marginal data density value greater than in the case of M_0 by more than six orders of magnitude. Among competing specifications with time-varying parameters the data substantially support the case where variability of unconditional moments can be described by an Almost Periodic function of the form (4) with different frequencies. The case with frequencies is also supported, however it does not receive as great data support as the APC(2)-GARCH(1,1) case.

Table 2 presents results of Bayesian estimation of parameters in AR(1)-GARCH(1,1) and APC(2)-GARCH(1,1) models (M_0 and M_2 respectively). We report the following posterior summaries: the mean ($E(.|y)$), the modal value ($Mod(.|y)$) and the standard deviation ($D(.|y)$) of the marginal distributions of parameters in both models. Posterior

inference about common parameters (degrees of freedom ν and AR-GARCH parameters i.e. subvectors μ, σ^2 in θ) remains almost the same in both models, as posterior summaries change only slightly after incorporating a deterministic function f into the volatility equation. The data support strong persistence of conditional variance, as the sum $\alpha_1 + \beta_1$ is located close to unity in case of model M_0 . The deterministic component in the volatility equation of model M_2 is empirically important, but it does not change posterior location of $\alpha_1 + \beta_1$ qualitatively. Also the posterior inference about tails of the conditional distribution of returns is almost the same in both models, leading to the conclusion that conditional normality is rejected even in the case of nonstationarity of the error term. On the other hand, moment existence as imposed according to Definition 2, is decisively supported, because posterior standard deviation makes a less than 4 degrees of freedom parameter improbable in view of the data. Generally, marginal posterior distributions of common parameters are very regular, symmetric and qualitatively the same in the case of both models. In contrast, the posterior distributions of model specific parameters in the APC(2)-GARCH(1,1) model are irregular, as expectation and modal value in those cases may be located in different areas of the parameter space. We see relative strong dispersion of parameters controlling amplitudes in (5), namely γ_{s1} , γ_{s2} , γ_{c1} and γ_{c2} . For frequency parameters ϕ_1 and ϕ_2 a more regular posterior distribution was obtained.

Figure 1 presents results of posterior inference about the length of the period p_i of a single cyclical component in (3) induced by posterior distribution of the frequency parameter ϕ_i . We show histograms of the length in years, according to the formula:

$$p_i = \frac{2\pi}{\phi_i 251},$$

assuming 251 trading days per year. We compare histograms between the best model M_2 , where f is defined as a sum of two different cyclical components, and slightly worse (but still enjoying strong data support) M_1 model, with f defined by a single cyclical component. In the case of the M_2 model we see that one component, of approximately 14 years, is very precisely identified, as the majority of the probability mass of the

posterior distribution of p_2 is located between the values of 13 and 15 years with the mode and median equal approximately 14 years. However, the posterior inference about the second cyclical component might be problematic due to a rather irregular and dispersed posterior distribution. The posterior median for parameter p_1 indicates the existence of a much longer cycle, with the length greater than 30 years. However, the probability mass of the posterior distribution is very dispersed, leaving considerably greater uncertainty about the length of this longer cycle in unconditional variance.

In contrast to model M_2 , we obtain a very irregular posterior distribution of the length of the cycle in model M_1 . According to the plot of the histogram of p_1 we see how problematic the inference about the possible cyclical behavior on unconditional variance may be, given APC(1)-GARCH(1,1) specification. Since the marginal posterior distribution of p_1 is bimodal, any inference based on posterior summaries leads to wrong conclusions. Posterior expectation is located in areas of domain completely precluded in view of the data. We see, that the mass of posterior distribution is located either around the value of $p_1 = 14$ and in the regions, where $p_1 > 35$. Formally in model M_1 the data support the length of the cycle equal approximately to 14 years, as the posterior median is slightly greater than 14.17. But taking into account regions identified by both local extremes we see that inference about the length of the cycle in unconditional variance and the qualitative profile of the cyclical effect is rather similar in both models M_1 and M_2 . Since in model M_1 there is only one cyclical component in function f , the irregular shape of posterior distribution of p_1 can be explained as clear data support for two different frequencies in the equation for unconditional variance, just like in model M_2 .

The effect of cyclicality in unconditional variance of the error term in (6) and its empirical importance is depicted in Figure 2. We present the plot with absolute returns and posterior means of unconditional variance calculated for each data point, with bounds covering the range of two posterior standard deviations. We also plotted the posterior mean of the unconditional variance in model M_0 , as a constant function at $E(V(\varepsilon_t)|y, M_0) = 0.9889$. Again, the data clearly support the variability of unconditional variance. The constancy of parameters is precluded since the changes of

variance in time are characterised by a considerably greater amplitude in the case of the M_2 model. The plots of $E(V(\varepsilon_t)|y, M_1)$ are characterised by fluctuations, strongly associated with long term changes in volatility in line with boom and bust periods on the US Stock Exchange. Another interesting aspect of the posterior inference about unconditional variance is connected with changes in the spread of posterior distributions of $V(\varepsilon_t)$. We see that it strongly declines in periods characterised by low volatility, but it becomes much greater in the case of periods of intensified volatility. This leaves much greater uncertainty about the possible deterministic profile of the cyclical component in the volatility equation in periods associated with crises when abnormal volatility is observed.

The main advantage of deterministic construct (3) is that the variability of unconditional variance in such a case can be easily fitted with economic interpretation. By definition, APC(F)-GARCH(1,1) processes are able to capture the long term cyclical behaviour of volatility. If some properties of the long term cycles in volatility are identified, one may ask the question whether those cycles are linked with changes in the economic activity of the real sector. Since the theory assumes a clear linkage between the real sector and the financial market, some crises exhibited by abnormal volatility should be associated with the decline of business activity. In Figure 3 we plotted again the results of posterior inference about unconditional variance in the best model (M_2) and additionally drew a plot representing time intervals attributed to recessions, as marked by NBER. Starting from the 50's, the NBER marked recession in the US economy on 10 occasions. For the postwar period the longest recession is linked with the global financial crisis which started in 2008. Also recessions in the mid 70's and at the beginning of the 80's were serious. In the vast literature concerning the empirical properties of the US business cycle, an interesting analysis was conducted by Chauvet and Potter (2001). According to the Bayesian analysis presented in that paper, the posterior expectation of time to wait until the next recession (if we are in recession currently) is equal approximately to 14 years. This fully corresponds to our posterior inference about frequencies ϕ_i and length in years p_i discussed above. However, according to Figure 3 we see how difficult it is to find a linkage between

long term volatility cycles and the business cycle. Only in the case of the recession in the mid 70's and during the dotcom crisis at the beginning of the 21st century, a visible increase in unconditional variance accompanies economic slowdown. Also some short recessions in the 50's coexist with a long-term but relatively small increase in unconditional volatility.

6 Conclusion

The main purpose of this paper was to investigate properties of a simple generalisation of the GARCH model that would enable to model long term features of volatility. Variability of unconditional moments was governed by a class of Almost Periodic (AP) functions, proposed by Corduneanu (1989). Since in our approach the unconditional second moment exhibits an almost periodic variability, the process can also be interpreted as a second order Almost Periodically Correlated (APC) stochastic processes; see Hurd and Miamee (2007).

We make a formal statistical inference, from the Bayesian viewpoint, about the cyclicity of volatility changes and present evidence in favour of the empirical importance of such an effect. The illustration was conducted on the basis of daily returns of the S&P500 index covering the period from the 18 January 1950 till the 7 February 2012.

According to Bayesian model comparison, the cyclical behavior of unconditional variance was strongly supported, making the AR(1)-GARCH(1,1) specification with constant parameters improbable. Among competing specifications, the greatest data support was received by the model where the time variability of the unconditional variance is described by a combination of two different cycles, with periods equal about 14 and 30 years. Those cycles were attributed to relatively different amplitudes, making the dynamic pattern of the unconditional volatility rather complex. Some evidence was obtained in favour of the linkage between the long term cyclical component in volatility and the business cycle for the postwar US economy.

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Table 1: Decimal logarithms of marginal data densities and decimal logarithms of the Bayes factor in favour of the best specification, for the GARCH(1,1) model and for the APC(F)-GARCH(1,1) model in the case of $F = 1, 2, 3$

Model	Number of frequencies	$\log_{10}p(y)$	$\log_{10}BF$
M_0	0 (GARCH(1,1) model)	0.29	6.68
M_1	1	3.94	3.04
M_2	2	6.98	
M_3	3	5.28	1.70

Table 2: Posterior modes ($Mod(\cdot|y)$), means $E(\cdot|y)$ and standard deviations ($D(\cdot|y)$) of parameters in APC(F)-GARCH(1,1) for the $F = 0$ (GARCH(1,1)) case and $F = 2$ (the best model)

Parameters	δ	ρ	α_0	α_1	β_1	ν	γ_{s1}	γ_{c1}	γ_{s2}	γ_{c2}	ϕ_1	ϕ_2
GARCH(1,1)												
$Mod(\cdot y)$	0.0486	0.1595	0.0044	0.0586	0.9366	5.93						
$E(\cdot y)$	0.0485	0.1592	0.0043	0.0587	0.9368	5.91						
$D(\cdot y)$	0.0057	0.0078	0.0007	0.0043	0.0045	0.28						
APC(2)-GARCH(1,1)												
$Mod(\cdot y)$	0.0488	0.1597	0.0077	0.0588	0.9296	5.96	-0.0377	0.3733	-0.3315	-0.2719	0.00090	0.0018
$E(\cdot y)$	0.0489	0.1596	0.0073	0.0589	0.9309	5.94	-0.1558	0.2738	-0.2434	-0.2589	0.00082	0.00022
$D(\cdot y)$	0.0057	0.0077	0.0012	0.0045	0.0052	0.29	0.2006	0.1262	0.1460	0.1398	0.00011	0.0042

Figure 1: Comparison between the M_1 and M_2 models of posterior inference about the length of the period (in years) p_i induced by frequency parameters ϕ_i according to the transformation $p_i = \frac{2\pi}{251\phi_i}$

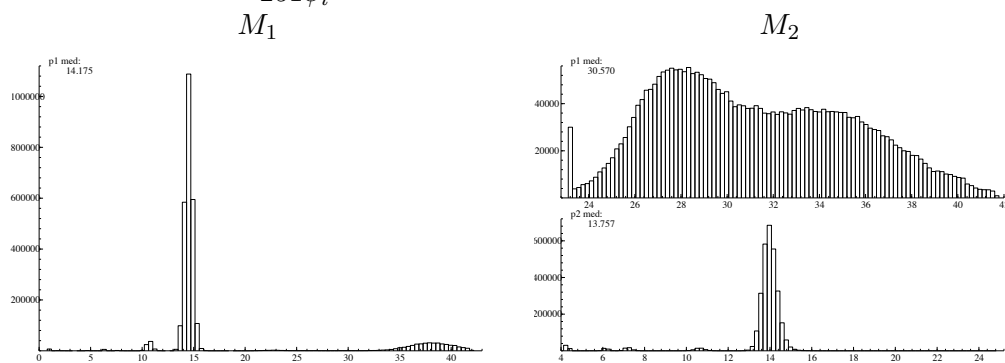


Figure 2: Posterior inference about unconditional variance of APC(2)-GARCH(1,1) for each data point

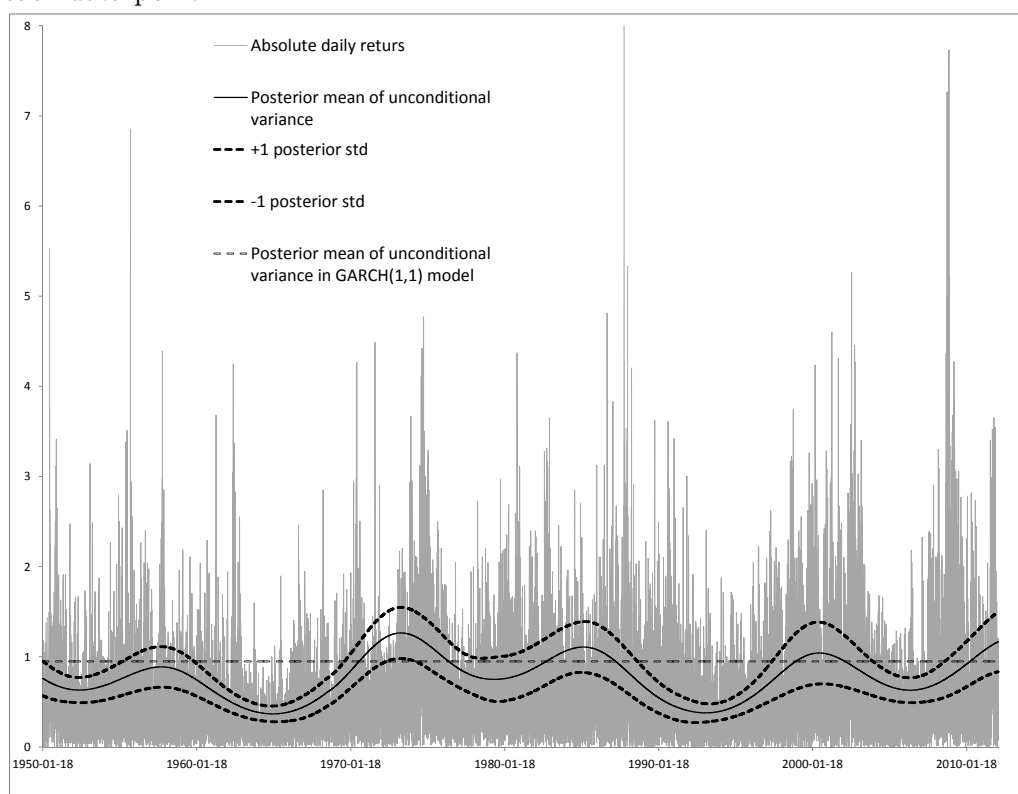


Figure 3: Coexistence of changes in unconditional variance and recession periods for postwar US economy marked by the National Bureau of Economic Research, NBER

