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A model for dependent defaults and pricing contingent claims with counterparty risk

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Abstract

This paper presents a new, intuitive but mathematically powerful model of dependent defaults and derives a general framework for pricing products whose values depend on credit correlation between the counterparty and the reference entity. The dependence framework is a natural extension of the Gaussian factor approach, which can be applied in the context of reduced form credit risk models, allowing i.a. for stochastic hazard and recovery rates. The prices of plain vanilla credit default swaps, first-to-default swaps and default swaptions are derived as particular examples.

Keywords: default correlation, counterparty risk, reduced form models

JEL classification codes: G12, G13

1 Introduction

This paper presents a new, simple and efficient way of pricing products whose values depend on credit correlation between the counterparty and the reference entity. A prominent example of such instruments is a credit default swap (CDS), i.e. a contract through which parties agree to exchange the credit risk of a given issuer. In a typical CDS, the protection buyer pays a periodic fee to the swap seller and in exchange receives compensation if the issuer undergoes a credit event, including e.g. failure to service outstanding debt obligations, debt restructuring or bankruptcy. Thus, it might seem straightforward that the protection buyer profits whenever the credit standing of the reference entity worsens, however in practice, the value of the swap will be affected also by the credit standing of the counterparty, and in particular by how the credit of the counterparty correlates with that of the reference entity. The case of positive dependence between the two is called "wrong-way risk", whereby the probability of default by the counterparty is high exactly at the time when the protection buyer's exposure to the counterparty – i.e. the present value of the swap – is high (one example would be banks selling CDSs on themselves or their own sovereign reference countries). In the extreme case when the default of the reference asset coincides with the default of the swap seller, the protection buyer would suffer considerable losses on both the underlying asset and the present value of the swap. Hence, to properly value a CDS contract – or more generally any contingent claim whose value depends on credit correlation between the counterparty and the reference entity – a model of credit correlation, or dependence, is needed.[1] This is underscored not only by the experiences of the 2008 financial crisis, and in particular the cases of Lehman Brothers, monoline insurers and AIG, but also by the recent regulatory trends under the heading of Basel III which stipulate that CVA risk, arising from changes in counterparties' credit spreads, should be identified and included in the calculation of economic capital (see also ECB, 2009 for an overview of post-crisis reflections on the pricing of counterparty risk in credit derivatives).

The need to include counterparty risk in the valuation of contingent claims has been well recognized in the literature. Hull and White (2001) provide what is considered today the first consistent and market-based methodology for valuing single-name and basket credit default swaps under counterparty risk. Their approach builds on Merton's (1974) structural credit risk model, whereby the creditworthiness of companies is defined by credit indices (or equity returns) which follow correlated Wiener processes, with default triggered by the first passage time of the firm's credit index to the threshold level, rather than some pre-specified time horizon as in the original formulation (cf. also Zhou, 1997). The model is calibrated to market data, in that the default barrier is chosen so that the default probabilities in the model are consistent with those implied from bond prices or CDS spreads. More recently, Walker (2006), Blanchet-Scalliet and Patras (2008), Brigo and Chourdakis (2009) as well as Buckley, Wilkens, and Chorniy (2011) have also proposed models for pricing derivatives in the presence of correlation between the counterparty and the reference entity, while Leung

[1] In what follows we shall use the words "correlation" and "dependence" interchangeably to express association (not necessarily linear) between random variables.

and Kwok (2005) and Brigo and Capponi (2010) derived CDS prices allowing in addition for correlation between the protection buyer, protection seller and the reference entity.

The counterparty risk models proposed so far fall roughly into one of the two broad strands of the literature. The first strand comprises models based on Merton's (1974) structural approach to credit risk where – as in Hull and White (2001) – default of an obligor is triggered when the market value of assets falls below a certain threshold level set by the level of debt at some maturity. Perhaps the greatest virtue of the Hull-White model, as well as other Merton-type approaches (see most recently Buckley, Wilkens, and Chorniy, 2011), is an intuitive setup and ease of calibration. The drawback, however, is that simultaneous default of a number of entities is jointly simulated by sampling from the multivariate normal distribution, which is well known to underestimate tail dependence (see e.g. McNeil, Frey, and Embrechts, 2005). Although this can be circumvented by enforcing a stronger dependence structure via a copula, the choice of a particular transformation of the distribution of the random vector would remain more or less arbitrary. Moreover, as pointed out by Mikosch (2006), for all their mathematical attractiveness, copulas are essentially static, suited to modeling spatial dependence, but inherently incapable of tackling complex space-time dependence structures which are at the heart of default time correlations.

The second major strand in the literature relates to the so called reduced form, or intensity-based, credit risk models, where default is not conditioned on the firm's asset values, but instead is considered a Poiosson-type event, occurring unexpectedly, according to some intensity process (see e.g. Lando, 2004, chap. 5 for an overview of the intensity-based approach). The general problem associated with standard reduced form models, as observed by Yu (2007), is that default correlation is induced by the correlation of default intensities – i.e. the default intensity of one party increases when the default of another party occurs – and default times themselves are independent conditional on the sample paths of default intensities (see e.g. Leung and Kwok, 2005 who price a CDS with counterparty risk in such a framework). Unfortunately, this means that such models can only produce default correlations that are very low and of the same order as default probabilities themselves (see e.g. Das, Duffie, Kapadia, and Saita, 2007 who show that empirically observed default correlations cannot be accounted for merely by correlated default intensities). With such a ceiling in place, it could prove challenging to calibrate the model to default correlations implied from quoted prices of various financial instruments, which in turn severely limits the scope for practical applications of such models. Several remedies have been proposed to this shortcoming (see e.g. Jarrow and Yu, 2001 or Gagliardini and Gouriéroux, 2003), but in general allowing for dependent defaults has rendered the models analytically rather challenging (see also Bielecki and Rutkowski, 2010, chap. 10, for a discussion of models with dependent intensities).

Thus, we propose a model that strives to address both limitations discussed above: is sufficiently flexible to yield a continuous interpolation between independence and comonotonicity of default times, while being intuitive and still tractable. Specifically, the model, which extends the idea presented in Gatarek (2010), is a generalization of Gaussian factor models and is similar in spirit to Giesecke (2003) and Elouerkhaoui (2006), but unlike the previous approaches which were suited mainly to the pricing of basket credit derivatives, our framework offers a general formula for the valuation of cash flows in the presence of counterparty risk, with particular formulas for CDS, first-to-default swaps, default swaptions etc. with stochastic recovery rates. Similarly to Giesecke (2003), we assume that a firm's default is determined by both firm-specific and systematic factors affecting all firms alike. However, we propose to think of a systematic factor as an increasing sequence of default times which allows preserving the martingale property. The model can accommodate any number of systematic factors and allows different obligors to have different correlation with each factor (sector), which is an improvement upon the Gaussian latent variable one factor model underlying the Creditmetrics approach and Basel regulations (Gordy 2003). The factors themselves can be chosen freely from the whole class of probability distributions on $\mathbb{R}_+$, allowing also for stochastic hazard rates. The dependence structure is imposed by the minimum function, so that – the factors being naturally interpreted as economy-wide, industry-specific, sectoral, regional etc. – a firm can default either due to own mismanagement or an adverse systematic shock – whichever hits it sooner.

The rest of the paper is structured as follows. Section 2 gives an overview of our framework for modeling default dependence. Section 3 derives the key valuation formulas along with some examples, while Section 4 concludes.

2 Modeling dependent defaults

In this section we present a general framework for modeling dependence that will be used later on for pricing contingent liabilities in the presence of counterparty risk. To set the stage, we start with a brief overview of the standard Gaussian factor model of dependence, then explain its deficiencies in modeling default correlation in a reduced-form framework and finally suggest a straightforward generalization based on the minimum function. For modeling purposes, we assume that all processes and variables are defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathcal{F}$ modeling the information flow and $\mathbb{P}$ being the risk-netrual (martingale) measure ensuring that all security prices discounted by the risk-free interest rate process are martingales.

2.1 Gaussian factor model

Let $X_1, X_2, ..., X_d$ be a family of Gaussian random variables. The most striaghtforward way to introduce dependence among them is to assume the following representation:

$$X_i = y_i Y_i + \sum_{j=1}^{N} z_{i,j} Z_j, \tag{1}$$

where $Y_1, .., Y_d$ and $Z_1, ..., Z_N$ are standard normal variables such that $\mathrm{cov}(Y_i, Y_k) = \mathrm{cov}(Z_j, Z_n) = \mathrm{cov}(Y_i, Z_j) = 0$ for $i \neq k$ and $j \neq n$, while $y_i, z_{i,j} > 0$ for $i = 1, 2, ..., d$ and $j = 1, ..., N$. By analogy with the capital asset pricing model (Sharpe, 1964), variables Y_i are called idiosyncratic risk factors and describe individual behavior of entities under investigation. Variables $Z_1, ..., Z_N$ are called systematic and describe N common factors driving the behavior of the whole population or parts of it containing more than one element. The idiosyncratic factor can be thought of as relating to firm-specific issues, as e.g. management quality, while the systematic factors capture the common link between entities, as it may result e.g. from operating in the same sector, selling the same type of product, etc.

A particularly interesting case is when all X_i are standard normal and there is only one systematic factor (the so called one factor model):

$$X_i = y_i Y_i + z_i Z. \tag{2}$$

Since $\mathrm{var}(X_i) = 1$, then $y_i^2 + z_i^2 = 1$, and coefficients z_i control the strength of the linear correlation between X_i and Z, with $\mathrm{corr}(X_i, X_j) = z_i z_j$.

The simple factor-type dependence framework described above arises naturally in the context of structural, or asset value models of credit risk inspired by Merton (1974) (see also Schönbucher, 2001 for a review). The structural models rest on the idea that corporate liabilities can be viewed as contingent claims on the assets of the firm, with default triggered by the fall in market value of assets below some debt threshold. If it is assumed that the firm's assets X_i are driven by a combination of idiosyncratic and systematic factors as in (1), then (using the fact that the sum of Gaussian variables is still Gaussian) probabilities of default and default correlations can be estimated which in turn allows to assess portfolio losses in a bottom-up fashion. The key assumption of the structural approach is that the value of assets and their volatility, the debt threshold, and thus also default time, are all based on a filtration that is observed by the market, even though in fact such information would normally be available only to the firm's management. O'Kane (2008) mentions two other important defficiencies of the structural model: (i) highly simplified capital structure, not allowing for a straightforward implementation of different levels of subordination and different maturities of the respective liabilities; (ii) unrealistic assumption of all liabilities being in the form of zero coupon bonds (coupon paying bonds can be viewed as compound options and thus significantly more complex computationally).

2.2 Ordered and idiosyncratic defaults in a reduced form framework

These shorcomings are to a large extent fixed in the so called reduced form credit risk models, suggested initially by Jarrow and Turnbull (1995), where it is assumed that that the modeler, much like the rest of the market, has only an incomplete knowledge of the firm, which seems more realistic and is thus recommended for pricing and risk management purposes (see also Jarrow and Protter, 2004 for a discussion of the information-based distinction of the two approaches). Also the modeling logic is completely reversed: while in the standard structural model the purpose was to find the probability of default conditioned on time and the firm's asset values, in the reduced form framework the objective is to model default time itself, which is considered a Poiosson-type event, occurring unexpectedly, according to some intensity process (see Lando, 2004, chap. 5 for an overview). The intensity, or hazard rate, can be interpreted as the instantaneous default probability and is mathematically expressed as:

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{\mathbb{P}(t < \tau \leq t + \Delta t \mid \tau > t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbb{P}(\tau \leq t + \Delta t) - \mathbb{P}(\tau \leq t)}{\Delta t (1 - \mathbb{P}(\tau \leq t))} = \frac{1}{1 - F(t)} \frac{dF(t)}{dt}, \tag{3}$$

where τ is a continuous random variable which measures the default time of a certain obligor and $F(t)$ is its distribution function. It then follows that $F(t) = 1 - \exp\left(-\int_0^t \lambda(s)ds\right)$ and in the particular case of a costant hazard rate we obtain the familiar exponential distribution with $F(t) = 1 - \exp(-\lambda t)$.

Note that even with the tools developed so far we can already represent two key insights of independent and co-monotonic defaults. To see this, suppose there are d obligors ("names") with default times $\tau_1, \ldots, \tau_d$ and consider two extreme types of economies: (i) an idiosyncratic economy, i.e. one where default times are affected only by idiosyncratic factors; and (ii) an ordered economy, where there are no idiosyncratic factors and there is only one systematic factor which amounts to default times $\tau_1, ..., \tau_d$ being ordered so that $\tau_i \leq \tau_j$ whenever $i \leq j$. Denote by $\tau_{first} = \min\{\tau_i : 1 \leq i \leq d\}$ the first default in either economy. If default times are completely independent, i.e. triggered by idiosyncratic factors, then

$$\mathbb{P}_{idio}(\tau_{first} \geq t) = \prod_{i=1}^{d}(1 - F_i(t)) = \exp\left(-\int_0^t \sum_{i=1}^{d} \lambda_i(s)ds\right). \tag{4}$$

In contrast, when defaults are ordered

$$\mathbb{P}_{order}(\tau_{first} \geq t) = 1 - F_1(t) = \exp\left(-\int_0^t \lambda_1(s)ds\right). \tag{5}$$

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Hence,

$$\frac{\mathbb{P}_{order}(t \leq \tau_{first} \leq t + dt \mid \tau_{first} > t)}{\mathbb{P}_{idio}(t \leq \tau_{first} \leq t + dt \mid \tau_{first} > t)} = \frac{\lambda_1(t)}{\sum_{i=1}^{d} \lambda_i(t)} < 1, \tag{6}$$

which leads to an intuitive conclusion that the probability of multiple defaults (driven by a systematic factor) is lower than that of single defaults. Note, however, that although defaults in the ordered economy are less frequent, their overall impact is more severe due to clustering. This also suggests that while counterparty risk (specifically, correlation between the reference credit and the counterparty) would not be a serious concern in the idiosyncratic economy, it could have a considerable impact on pricing contingent claims in the ordered economy, as e.g. buying protection might entail taking on wrong way risk.[2]

Clearly, no economy is either completely idiosyncratic or completely ordered, so what we need is a continuous interpolation between the two cases discussed above. Unfortunately, the convenient representation (1) cannot be readily used to describe dependence in a reduced form framework, as it relied crucially on the fact that the sum of Gaussian variables remained Gaussian – a property that does not in general hold for other distributions. One way to get around this problem would be to apply a copula transformation of the risk factor distributions, but such a trick would be artificial and would entail breaking the natural correspondence between the described reality and suitable mathematical tools. We propose what seems to be a more elegant solution, namely to preserve the risk factor decomposition from (1), but use the minimum function instead of the sum. Since in the reduced form framework default is a one-time event – i.e. once a name goes bust it can never default again – it is intuitive to assume that default occurs whenever an obligor is hit by one of the shocks for the first time (this is also the assumption behind the familiar Marshall-Olkin copula used by Giesecke, 2003).

2.3 A general model of dependence

By analogy with the Gaussian factor model, let $\{1, \ldots, d\}$ be the set of all obligors in the economy with default times $X_1, \ldots, X_d$ driven by two sets of risk factors: idiosyncratic and systematic. To proceed we first need to formalize the definition of a systematic factor

[2]This observation has another interesting implication. Consider a CDO made up of names $\{1, ..., d\}$. Investors in the senior tranche would prefer the economy to be idiosyncratic – they can afford to suffer even a few uncorrelated defaults, but they fear hitting a cluster in an ordered economy. In contrast, investors in the equity tranche would prefer an ordered economy: they blow up irrespective of whether a single name defaults or a whole cluster, but – like the blindfolded cat in JP Morgan's parable – with clusters they can at least find a way among the "mousetraps."

Definition 1. A family of random variables Z_i, $1 \leq i \leq d$, with given distributions is called a systematic factor if and only if there exists a permutation Φ of the set $\{1, 2, \ldots, d\}$ such that the sequence $Z_{\Phi(i)}$ is increasing i.e. $Z_{\Phi(i)} \leq Z_{\Phi(i+1)}$ for $i = 1, ..., d$.

We say that factors Z and Y are independent if all pairs of random variables Z_i and Y_j $(i, j = 1, ..., d)$ in the respective families are independent. Definition (1) brings an important change: while in (1) a systematic factor was a single random variable, from now on it will be understood as a family of random variables which can be put in increasing order. Although such a change may seem unintuitive at first, in fact it is the only way to allow different obligors to have different correlation with the systematic factors, without sacrificing the martingale property. To see this, suppose *a contrario* that each X_i is represented in the following way:

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$$X_i = \min \{Y_i, z_{i,1}Z_1, ..., z_{i,N}Z_N\}, \tag{7}$$

where $Y_1, ..., Y_d$ and $Z_1, ..., Z_N$ are independent positive random variables and $z_{i,j}$ are positive constants. In line with (1), random variables Y_i can be interpreted as idiosyncratic factors and Z_j as systematic risk factors, with hazard rates $\lambda_1(t), \ldots, \lambda_N(t)$. Note, that if $z_{k,j} < z_{i,j}$, for some $i, k \in \{1, ..., n\}$, $j \in \{1, ..., N\}$, then $z_{i,j}Z_j = \phi(z_{k,j}Z_j)$ for a deterministic and increasing function $\phi(x) = \frac{z_{i,j}}{z_{k,j}}x$, which guarantees that $z_{k,j}Z_j < z_{i,j}Z_j$ in all possible "states of the world" and $z_{i,j}Z_j$ can be fully predicted based on $z_{k,j}Z_j$. This is an extreme case of co-monotonicity, particularly unwelcome in modeling default times, since – apart from the case of parent companies and their subsidiaries – it is in practice unintuitive to assume that the default time of one company is deterministically followed by the default of another company. Brigo and Morini (2010) go even so far as to call such an assumption: "a feature with scarce financial meaning that can lead to misleading results." Such co-monotonicity implies in particular that for a default indicator process defined as $N_{i,j}(t) = \mathbb{1}_{\{z_{i,j}Z_j > t\}}$, $N_{k,j}(t) \geq N_{i,j}(t)$ and the process $N_{k,j}(t)$ predicts $N_{i,j}(t)$. It follows that $N_{i,j}(t) + \int_0^t N_{i,j}(s)\frac{1}{z_{i,j}}\lambda_j(s)ds$ is not a martingale for all i with respect to the expanded filtration $F_{i,j}(t) = \sigma(N_{i,j}(t), i = 1, ..., d)$, which in turn implies that $\lambda_j(t)/z_{i,j}$ are not hazard rates, which is counter-intuitive. Therefore, the set $\{z_{i,j}Z_{i,j} : 1 \leq i \leq d\}$ cannot represent a systematic factor.

Systematic factors in the sense of Definition 1 can be constructed in the following way. Let $\tau_1, ..., \tau_d$ be a sequence of positive random variables conditionally independent upon hazard rate processes $\lambda_1(t), ..., \lambda_d(t)$. Let $\Phi(\cdot)$ be a permutation of $\{1, ..., d\}$ such that $\tau_{\Phi(1)} \leq ... \leq \tau_{\Phi(d)}$ (i.e. $\tau_{\Phi(i)}$ is the i-th order statistic). Then the family $Z_i = \min\{\tau_{\Phi(j)} : \Phi^{-1}(i) \leq j \leq d\}$ is a systematic factor. Indeed, the sequence $Z_{\Phi(i)} = \min\{\tau_{\Phi(j)} : i \leq j \leq d\}$ is increasing. Moreover, by the Doob-Meyer theorem, the decreasing one-jump event indicator process $N_i(t) = \mathbb{1}_{\{Z_i > t\}}$ associated with each Z_i can be compensated so that $N_i(t) + \int_0^t N_i(s)\xi_i ds$ is a martingale and hence $\xi_i(t) = \sum_{j=i}^d \lambda_{\Phi(j)}(t)$ is the hazard rate (see e.g. Schönbucher, 2003, chap. 4). Obviously, the above construction can be extended to any number of systematic factors which lays the ground for our model of dependent defaults.

Definition 2. (Dependent defaults) Let $\{1,...,d\}$ be the set of all obligors and let $U = \left\{Y_i, Z_i^1, \ldots, Z_i^N : 1 \le i \le d\right\}$ be the set of positive random variables describing default times of all risk factors, both idiosyncratic, Y_i, and systematic $Z_i^1, \ldots, Z_i^N$ (in the sense of Definition 1). Let $\lambda_\tau(t)$, $\tau \in U$ be a family of continuous stochastic processes, not necessarily independent, forming hazard rates for risk factors τ. Then we define dependent default times as

$$X_i = \min\left\{Y_i, Z_i^1, \ldots, Z_i^N\right\} = \min_{\tau \in U(i)} \tau, \tag{8}$$

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where

$$U(i) = \{Y_i, \xi^j_{\Phi_j(k)} : \Phi_j^{-1}(i) \le k \le d, 1 \le j \le N\} \subset U \tag{9}$$

and $\Phi_j(\cdot)$ are the permutations ordering the sequences $(\xi^j_{i.})_{i=1}^d$, $j = 1,...,N$.

It follows from Definition 2 that

$$\mathbb{1}_{\{\tau > t\}} + \int_0^t \mathbb{1}_{\{\tau > s\}} \lambda_\tau(s) ds \tag{10}$$

is a martingale. Furthermore, since $\mathbb{1}_{\{X_i > t\}} = \prod_{\xi \in U(i)} \mathbb{1}_{\{\xi > t\}}$, where the variables ξ are order statistics of systematic factors τ (we omit subscripts to simplify notation), then assuming that all risk components τ are conditionally independent upon hazard rate processes,

$$\mathbb{1}_{\{Y > t\}} + \int_0^t \mathbb{1}_{\{X_i > s\}} \sum_{\tau \in V} \lambda_\tau(s) ds, \tag{11}$$

is also a martingale for any subset $V \subseteq U$ and $Y = \min_{\tau \in V} \tau$.

The following example should help clarify the idea behind our model of dependent defaults.

Example 1. Consider an economy which consists of only three obligors $\{1, 2, 3\}$ subject to two systematic factors $\{\tau_1, \tau_2, \tau_3\}$ and $\{\vartheta_{1,}, \vartheta_2, \vartheta_3\}$ (to simplify the exposition assume there are no idiosyncratic factors). Suppose $\Phi_1 : \{1, 2, 3\} \to (3, 2, 1)$ and $\Phi_2 : \{1, 2, 3\} \to (2, 3, 1)$ are permutations of $\{1, 2, 3\}$ such that $\tau_{\Phi_1(1)} \le \tau_{\Phi_1(2)} \le \tau_{\Phi_1(3)}$ and $\vartheta_{\Phi_2(1)} \le \vartheta_{\Phi_2(2)} \le \vartheta_{\Phi_2(3)}$. In other words, when τ hits, obligor "3" is the first to suffer, followed by "2" and "1". With ϑ, on the other hand, problems first hit "2", then "3" and finally "1", whereby of course defaults triggered by each factor can occur at different times. Now, suppose we want to identify the default time of obligor "2". Since $\Phi_1(2) = 2$ and $\Phi_2(2) = 1$, applying (9), yields $U(2) = \{\xi_{\Phi_j(k)} : \Phi_j^{-1}(2) \le k \le d, j = 1, 2\} = \left\{\tau_{\Phi_1(2)}, \tau_{\Phi_1(3)}, \vartheta_{\Phi_2(1)}, \vartheta_{\Phi_2(2)}, \vartheta_{\Phi_2(3)}\right\}$. Since $\tau_{\Phi_1(2)} \le \tau_{\Phi_1(3)}$ and $\vartheta_{\Phi_2(1)} \le \vartheta_{\Phi_2(2)} \le \vartheta_{\Phi_2(3)}$, default time of "2" is given by $X_2 = \min_{\xi \in U(2)} \xi = \min\{\tau_{\Phi_1(2)}, \vartheta_{\Phi_2(1)}\}$. The procedure of finding default times is also represented schematically in Figure 1.

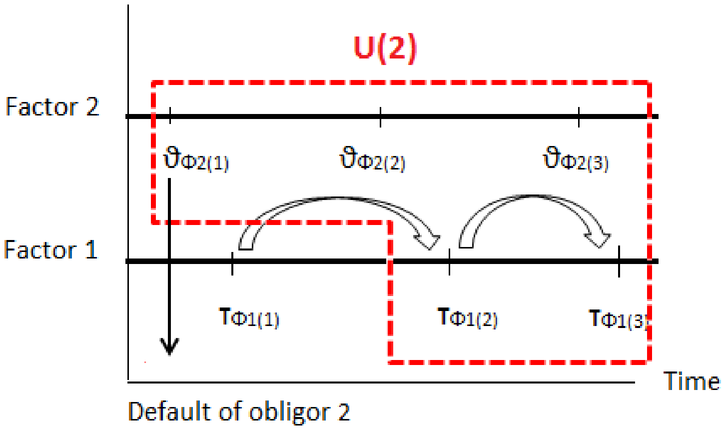
We have seen that default of obligor i is ultimately triggered by the lowest of the i-th order statistics $\tau_{\Phi_j(i)}$, $j = 1, ..., N$. Note that, as shown in Figure 1, both the ordering of default times and time lags between them can vary across factors (hence the N different permutations). Such a representation reflects an important economic observation that while for a systematic factor j a default of obligor A may trigger the default of obligor B, for the factor k the order may be reversed, with problems of B preceding a default of A, quite possibly with a different time lag than before. More generally, as stressed above, the redefinition of a systematic factor as an ordered family of random variables (rather than a single random variable as in (1) and (7)) allows us to incorporate obligors' different dependence on the systematic factors while preserving the martingale property (10).

We can now prove the following key fact.

Proposition 1. *For all $t < T$ and $\tau, \lambda_\tau(t), U(i), \mathbb{1}_{\{\tau>t\}}, \mathbb{1}_{\{X_i>t\}}$ defined above, conditional survival probabilities under the natural filtration $\mathcal{F}_t$ are given by:*

$$\mathbb{P}(\tau > T \mid \mathcal{F}_t) = \mathbb{E}\left(e^{-\int_t^T \lambda_\tau(s)ds} \mid \mathcal{F}_t\right) \mathbb{1}_{\{\tau>t\}}$$
$$\mathbb{P}(X_i > T \mid \mathcal{F}_t) = \mathbb{E}\left(e^{-\int_t^T \sum_{\tau\in U(i)} \lambda_\tau(s)ds} \mid \mathcal{F}_t\right) \mathbb{1}_{\{X_i>t\}}.$$

Figure 1: A schematic representation of an economy with three obligors and two systematic factors.



Proof. Denote $M(t) = \mathbb{E}\left(\exp\left(-\int_t^T \lambda_\tau(s)ds\right) \mid \mathcal{F}_t\right)$. Then, using Itô's product rule,

$$d\left(M(t)\mathbb{1}_{\{\tau>t\}}\right) = M(t)d\mathbb{1}_{\{\tau>t\}} + \mathbb{1}_{\{\tau>t\}}dM(t) + dM(t)d\mathbb{1}_{\{\tau>t\}}. \tag{12}$$

By assumption, $dM(t)d\mathbb{1}_{\{\tau>t\}} = 0$ and since $dM(t) = M(t)\lambda_\tau(t)dt$ we get

$$d\left(M(t)\mathbb{1}_{\{\tau>t\}}\right) = M(t)\left(d\mathbb{1}_{\{\tau>t\}} + \lambda_\tau(t)\mathbb{1}_{\{\tau>t\}}\right) \tag{13}$$

Hence, $M(t)\mathbb{1}_{\{\tau>t\}}$ is a martingale. Since $M(T) = 1$, we can write

$$\mathbb{P}\left(\tau > T \mid \mathcal{F}_t\right) = \mathbb{E}\left(\mathbb{1}_{\{\tau>T\}} \mid \mathcal{F}_t\right) = \mathbb{E}\left(M(T)\mathbb{1}_{\{\tau>T\}} \mid \mathcal{F}_t\right), \tag{14}$$

Finally, using the martingale property:

$$\mathbb{P}\left(\tau > T \mid \mathcal{F}_t\right) = \mathbb{E}\left(M(T)\mathbb{1}_{\{\tau>T\}} \mid \mathcal{F}_t\right) = M(t)\mathbb{1}_{\{\tau>t\}}. \tag{15}$$

An analogous reasoning proves the second equality. □

The general framework developed above paves the way for valuing contingent liabilities in a setting where the credit quality of the reference asset and that of the counterparty are driven by the same systematic factors. That is indeed what we address in the following section.

3 Valuing contingent claims with counterparty risk

The basic setup in this section draws on that from subsection 2.3 but, apart from the d obligors with default times $X_1, ..., X_d$, we introduce also two types of assets: the risk-free money market account B and a financial asset A. Assume the money market account accumulates an instantaneous interest rate given by the continuous process $r(t)$ so that $B(t) = \exp\left(\int_0^t r(s)ds\right)$ and the associated discount factor is $D(t) = \exp\left(-\int_0^t r(s)ds\right)$. Assume also that $A(t)$ admits the following representation:

$$dA(t) = A(t)(dW(t) - \alpha(t)dt), \tag{16}$$

where α is a continuous stochastic process and W is a martingale with zero quadratic covariation with $\mathbb{1}_{\{X_i>t\}}$. A contingent claim will be here understood in the most general sense as any random variable defined on Ω (clearly, this includes random variable whose payoffs depend on $A(t)$). To facilitate exposition, and in line with e.g. Brigo and Masetti (2005), we will be interested in the "unilateral risk of default" where only the risk of the counterparty is analyzed while the investor is considered risk free, as an approximation of cases where the investor has a much higher credit quality than the swap seller. It should be stressed however, that our framework can quite simply be extended to cover also the case where both parties can default. With this in mind, the pricing of contingent claims consists in calculating the risk neutral (martingale) expectation of the discounted cash flows of the derivatives from t to T. Thus, the net present value $PV(t,T)$ of a generic derivative is given by:

$$PV(t,T) = \mathbb{E}\left(D(T)A(T)\mathbb{1}_{\{X_i>t\}} \mid \mathcal{F}_t\right). \tag{17}$$

Note that the expectation $\mathbb{E}(\cdot \mid \mathcal{F}_t)$ is taken with respect to market information up to time t. It turns out that (17) can be re-expressed in the following convenient way.

Proposition 2. *Let $PV(t,T)$ be the net present value of cash flows of a contingent claim subject to the credit risk of counterparty i. The following relation holds:*

$$PV(t,T) = \mathbb{E}\left(\exp\left(-\int_t^T \alpha(s) + r(s) + \sum_{\tau\in U}\lambda_\tau(s)ds\right) \mid \mathcal{F}_t\right) A(t)D(t)\mathbb{1}_{\{X_i>t\}}.$$

Proof. The proof is analogous to the proof of Proposition 1. Denote

$$M(t) = \mathbb{E}\left(\exp\left(-\int_t^T \alpha(s) + r(s) + \sum_{\tau\in U}\lambda_\tau(s)ds\right) \mid \mathcal{F}_t\right) A(t)D(t). \tag{18}$$

Using Itô's product rule and the fact that $\left\langle W(t), \mathbb{1}_{\{X_i>t\}}\right\rangle = 0$, we find that

$$dM(t) = d\mathbb{E}\left(e^{-\int_t^T \alpha(s)+r(s)+\sum_{\tau\in U}\lambda_\tau(s)ds} \mid \mathcal{F}_t\right) A(t)D(t) +$$

$$+ d(A(t)D(t))\mathbb{E}\left(e^{-\int_t^T \alpha(s)+r(s)+\sum_{\tau\in U}\lambda_\tau(s)ds} \mid \mathcal{F}_t\right) + \tag{19}$$

$$+ d\mathbb{E}\left(e^{-\int_t^T \alpha(s)+r(s)+\sum_{\tau\in U}\lambda_\tau(s)ds} \mid \mathcal{F}_t\right) d(A(t)D(t)), \tag{20}$$

which in view of $dD(t)dA(t) = 0$ reduces to:

$$\begin{aligned} dM(t) = d\mathbb{E}&\left(e^{-\int_t^T \alpha(s)+r(s)+\sum_{\tau\in U}\lambda_\tau(s)ds} \mid \mathcal{F}_t\right) A(t)D(t) + \\ &+ \left(dA(t)D(t) + A(t)dD(t)\right)\mathbb{E}\left(e^{-\int_t^T \alpha(s)+r(s)+\sum_{\tau\in U}\lambda_\tau(s)ds} \mid \mathcal{F}_t\right) \end{aligned} \tag{21}$$

and ultimately to

$$dM(t) = M(t)\left(dW(t) + \sum_{\tau\in U}\lambda_\tau(t)dt\right). \tag{22}$$

Thus, using once again the product rule,

$$d\left(M(t)\mathbb{1}_{\{X_i>t\}}\right) = M(t)\left(dW(t)\mathbb{1}_{\{X_i>t\}} + \mathbb{1}_{\{X_i>t\}}\sum_{\tau\in U}\lambda_\tau(t)dt + d\mathbb{1}_{\{X_i>t\}}\right). \tag{23}$$

Hence, $M(t)\mathbb{1}_{\{X_i>t\}}$ is a martingale and since $M(T) = A(T)D(T)$, (17) finally becomes:

$$PV(t,T) = \mathbb{E}\left(M(T)\mathbb{1}_{\{X_i>t\}} \mid \mathcal{F}_t\right) = M(t)\mathbb{1}_{\{X_i>t\}}, \tag{24}$$

which concludes the proof. □

3.1 Plain vanilla CDS

Proposition 2 provides a general framework for the valuation of a broad class of contingent claims in the presence of counterparty risk. The valuation of credit derivatives – e.g. credit default swaps – can be derived as a corollary. As briefly explained in the introduction, a credit default swap (CDS) is a contract that facilitates the transfer of credit risk on some reference asset (e.g. corporate or sovereign bond) from one party to another (see e.g. O'Kane and Turnbull, 2003 for a primer on credit default swaps). Under the terms of the transaction, protection buyer makes regular, usually quarterly, payments p to the protection seller, the size of which is quoted as CDS spread and is paid on the face value of the asset under protection – the so called fixed or premium leg. The premium payments are made until the transaction matures or the reference credit defaults. Typically, and this is what we are going to assume below, payments also cease once the protection seller defaults, however it is also possible that payments are made and added to the swap seller's bankruptcy estate. Hence, the premium payment on each of the payment dates $t_1, t_2, ..., t_N = T$ equals $p\delta_n\mathbb{1}_{\{X_i>t_n\}}\mathbb{1}_{\{X_j>t_n\}}$, where δ_n is the day count fraction for the period $[t_{n-1}, t_n]$. The contingent (or protection) leg of the swap consists of a single payment made by the protection seller in case of default of the reference issuer before maturity time T. The protection leg can be settled either in cash or, more typically, by physical delivery, whereby the protection buyer delivers face value of the defaulted asset and receives a payment equal to its face value. In either case, protection buyer's payoff at default is equal to the notional multiplied by $(1-R)$, R being the recovery rate on the defaulted asset. Putting this together, the value of a CDS contract to the protection buyer is equal to the difference between the present value of the contingent leg and the fixed leg, taking into account the dependence of both the issuer's (name i) and the protection seller's (name j) credit on risk factors:

$$\begin{aligned} CDS(t,T) = (1-R)\mathbb{E}\left(\int_t^T D(s)\mathbb{1}_{\{X_j>s\}} d\mathbb{1}_{\{X_i>s\}} \mid \mathcal{F}_t\right) - \\ -\mathbb{E}\left(\sum_{n=1}^N p\delta_n D(t_n)\mathbb{1}_{\{X_i>t_n\}}\mathbb{1}_{\{X_j>t_n\}} \mid \mathcal{F}_t\right), \end{aligned} \tag{25}$$

The generic formula can be restated in the following way.

Proposition 3. *Let name i (with default time X_i) be the issuer of a risky bond, j (with default time X_j) the seller of protection on i's debt, p denote the premium (CDS spread) paid for protection on coupon periods set by dates $t_1, t_2, ..., t_N = T$ and R be the recovery rate on i's debt. Then the net present value of the CDS to the protection buyer is given by:*

$$\begin{aligned} CDS(t,T) = (1-R)\mathbb{E}\left(\int_t^T G(s)\sum_{\tau\in V}\lambda_\tau(s)ds \mid \mathcal{F}_t\right) D(t)\mathbb{1}_{\{X_i>t\}}\mathbb{1}_{\{X_j>t\}} - \\ -\mathbb{E}\left(\sum_{n=1}^N p\delta_n \exp\left(-\int_0^{t_n} r(s) + \sum_{\tau\in U(i)\cup U(j)} \lambda_\tau(s)ds\right) \mid \mathcal{F}_t\right) D(t)\mathbb{1}_{\{X_i>t\}}\mathbb{1}_{\{X_j>t\}}, \end{aligned}$$

where $V = U(i)\setminus U(j)$ and $G(s) = \exp\left(-\int_t^s r(u) + \sum_{\tau\in V}\lambda_\tau(u)du\right)$.

Proof. The proof is analogous to that of Proposition 2. □

The fair value of the CDS spread (premium) is found by setting the value of the contract at inception to zero, i.e. $CDS(0,T) = 0$. Note that although the CDS valuation formula given above features a constant recovery rate, it can be easily generalized for stochastic recovery rates. Indeed, this follows from Proposition 2 by letting $A(t)$ be the stochastic recovery rate process.

Proposition 4. *Let the assumptions of Proposition 3 hold. Assume also the loss given default $L(t)$ on the bond issued by obligor i is a stochastic process satisfying the following differential equation: $dL(t) = L(t)(dW(t) - l(t)dt)$ where $l(t)$ is a continuous stochastic process and W is a martingale with zero quadratic covariation with $\mathbb{1}_{\{X_i>t\}}$. Then the protection leg of the CDS on i sold by j is given by:*

$$\begin{aligned} PL(t,T) = \mathbb{E}\left(\int_t^T D(s)L(t)\mathbb{1}_{\{X_j>s\}} d\mathbb{1}_{\{X_i>s\}} \mid \mathcal{F}_t\right) = \\ = \mathbb{E}\left(\int_t^T G(s)\sum_{\tau\in V}\lambda_\tau(s)ds \mid \mathcal{F}_t\right) D(t)L(t)\mathbb{1}_{\{X_i>t\}}\mathbb{1}_{\{X_j>t\}}, \end{aligned}$$

where $V = U(i) \setminus U(j)$ *and* $G(s) = \exp\left(-\int_t^s r(u) + l(u) + \sum_{\tau \in V} \lambda_\tau(u)du\right)$.

Proof. The proof is analogous to that of Proposition 2. $\square$

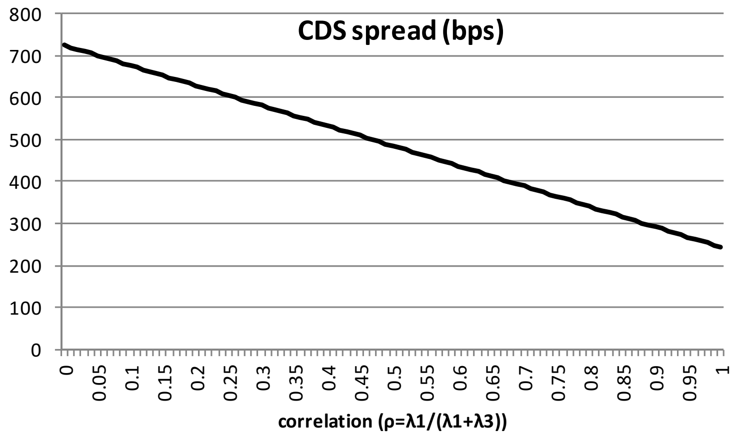
To illustrate how default correlation – i.e. dependence on systematic factors – affects counterparty risk and alters the valuation of a CDS consider the following example.

Example 2. Let A be the counterparty selling protection against the default of a more risky issuer B. Assume the CDS has maturity of $T = 10$ years and premium payments are made on a quarterly basis. Assume also a deterministic 40% recovery rate on B's debt, $R = 0.4$, and a one-factor version of the dependence model introduced in Definition 2, whereby idiosyncratic shocks are represented by exponential random variables Y_1 and Y_2 with parameters $\lambda_1, \lambda_2 > 0$. The systematic factor is constructed with the use of two exponential random variables τ_1, τ_2, with parameters ξ_1, ξ_2, such that $\xi_2 < \xi_1$. Thus, for $\Phi : (1,2) \to (2,1)$, $\tau_{\Phi(1)} \le \tau_{\Phi(2)}$. Then the sequence $Z_{\Phi(1)} = Z_2 = \min\{\tau_1, \tau_2\}$ and $Z_{\Phi(2)} = Z_1 = \tau_1$ is obviously increasing. In line with Definition 2, correlated default times are given by $X_A = \min U(A) = \min\{Y_1, \tau_1\}$ and $X_B = \min U(B) = \min\{Y_2, \min\{\tau_1, \tau_2\}\}$. Finally, the hazard rates of A and B are:

$$\begin{aligned} \lambda_A =& \lambda_1 + \xi_1 \\ \lambda_B =& \lambda_2 + \xi_1 + \xi_2 \end{aligned} \tag{26}$$

With these assumptions, Proposition 3 yields (FL being the fixed leg and PL the pro-

Figure 2: CDS spread as a function of default correlation ($T = 10$, $\lambda_i = 0.001$, $r = 2\%$, $R = 40\%$).



tection leg):

$$\begin{aligned} FL =& -p \sum_{t=1}^{40} e^{-r0.25t} e^{-(\lambda_A+\lambda_B)0.25t} = e^{-(\lambda_A+\lambda_B+r)0.25}\left(1 - e^{-(\lambda_A+\lambda_B+r)10}\right) \\ PL =& 0.6 \int_0^{10} e^{-rt} e^{-(\lambda_1+\xi_1)t}(\lambda_2+\xi_2)e^{(\lambda_2+\xi_2)t}dt = \frac{(\lambda_2+\xi_2)\left(1 - e^{-(r+\lambda_1+\lambda_2+\xi_1+\xi_2)10}\right)}{r+\lambda_1+\lambda_2+\xi_1+\xi_2} \end{aligned} \tag{27}$$

Note that when $\xi_1 = 0$ the defaults of A and B are totally independent. In contrast, when $\lambda_1 = 0$, so that $\lambda_A = \xi_1$ the defaults are co-monotonic. Thus, a plausible measure of default correlation, which provides an interpolation between perfect dependence and independence, is: $\rho = \xi_1/(\lambda_1 + \xi_1)$ (analogous measure of default correlation is used by Giesecke, 2003). Assume that probabilities of default of both A and B are given by the market and hence λ_A and λ_B are fixed. In such circumstances, $FL(0,10)$ is fixed as well, which squares with the intuition that the traded credit risk is contained in the protection leg of the swap. We can now investigate the impact of default correlation on the CDS spread:

$$p = \frac{1}{FL(0,10)} PL(0,10) = \frac{0.6}{FL(0,10)} \frac{\lambda_B - \lambda_A \rho}{r + \lambda_B + \lambda_A(1-\rho)} \left(1 - e^{-(r+\lambda_B+\lambda_A(1-\rho))10}\right), \quad (28)$$

and using the fact that $\exp(x) \approx 1 + x$, $p \approx \frac{10}{FL(0,10)} (\lambda_B - \lambda_A \rho)$. Thus, the CDS spread is a decreasing function of default correlation (Figure 2) and an increasing function of the tenor.

3

3.2 First-to-default swaps

Our approach to modeling dependent defaults comes in very handy also in the context of pricing correlation products such as default baskets. Basket default swaps are similar in nature to ordinary credit default swaps, except they reference a whole basket, or group, of obligors, rather than one specific entity. The payment of the protection leg in a basket CDS (and termination of the fixed leg) is the nth – typically first, fifth or tenth – default in a basket (see e.g. O'Kane, 2008 for an extensive overview of single- and multi-name credit derivatives). Basket CDSs are an obvious hedging instrument for diversified credit portfolios, where buying protection on, say, the first default can be more efficient than buying CDSs on each name in the portfolio individually. In turn, from the perspective of protection seller, a first-to-default swap (FtD) is attractive as it leverages the spread premium relative to a single-credit asset paying a comparable spread, thus allowing to pocket a high spread while minimizing the actuarial risk (cf. O'Kane, 2008, pp. 227-229).

As with a standard CDS, the value of a first-to-default swap in the presence of counterparty risk (from the perspective of the protection buyer) is equal to the difference between the present value of the contingent leg and the fixed leg:

$$\begin{aligned} FDS(t,T) = (1-R)\mathbb{E}\left(\int_t^T D(s) \mathbb{1}_{\{X_j > s\}} d \prod_{i \in FtD} \mathbb{1}_{\{X_i > s\}} \mid \mathcal{F}_t \right) - \\ - \mathbb{E}\left(\sum_{n=1}^{N} pD(t_n) \prod_{i \in FtD \cup \{j\}} \mathbb{1}_{\{X_i > t_n\}} \mid \mathcal{F}_t \right), \end{aligned} \quad (29)$$

where $\{1, 2, ..., d\}$ is the basket of obligors, $FtD \subset \{1, 2, ..., d\}$, p is the spread, R the recovery rate and $t_1, t_2, ..., t_N = T$ denote the coupon periods. Formula (29) can be restated in the following way allowing for a stochastic loss given default.

Proposition 5. *Using the notation and assumptions introduced above, the net present value of a first-to-default swap to the protection buyer is given by:*

$$FDS(t,T) = \mathbb{E}\left(\int_t^T G(s) \sum_{\tau \in V^-} \lambda_\tau(s)ds \mid \mathcal{F}_t\right) D(t)L(t) \prod_{i \in FtD \cup \{j\}} \mathbb{1}_{\{X_i > t\}} -$$

$$- \mathbb{E}\left(\sum_{n=1}^{N} p \exp\left(-\int_0^{t_n} r(s) + \sum_{\tau \in U(i) \cup U(j)} \lambda_\tau(s)ds\right) \mid \mathcal{F}_t\right) D(t) \prod_{i \in FtD \cup \{j\}} \mathbb{1}_{\{X_i > t\}},$$

where $V = \bigcup_{i \in FtD} U(i) \cup U(j)$, $V^- = \bigcup_{i \in FtD} U(i) \backslash U(j)$ *and* $G(s) = e^{-\int_t^s r(u) + l(u) + \sum_{\tau \in V} \lambda_\tau(u)du}$.

Proof. The proof is analogous to that of Proposition 2. □

3

3.3 Default swaptions

As a final application, we show how to value options on CDSs in the presence of counterparty risk. We rely heavily on Schönbucher (2004) who first proved that such options can be priced using the famous Black (1976) formula by expressing the option payoff in terms of a defaultable numeraire asset (the change-of-numeraire technique has been applied in this context also by Jamshidian, 2004). A credit default swaption may be thought of as an explicit option on a CDS – i.e. an option to buy protection on a reference asset at a specified spread – or an option to extend an existing CDS contract. We consider only the so called European knockout swaptions which give the option holder the right to buy protection only on one specific date (expiry) and cancel automatically with no payments if there is a credit event before expiry (see e.g. O'Kane, 2008 for a general overview of credit swaptions). The knockout property is crucial: the option owner is long protection forward, so ideally he would like the credit of the reference name to deteriorate and the spread to widen relative to the strike price, but he would not want default as such to occur before expiry, i.e. before he had a chance to exercise the option. Much like plain vanilla CDSs, credit swaptions are also highly sensitive to counterparty risk, and default time correlation between the protection seller and the reference entity will have a significant impact on the option price.

Since the underlying instrument in credit swaptions is a forward starting CDS, we can build on the framework developed above for plain vanilla CDSs. As before, let i be the issuer of a risky bond and j the seller of protection on i with an embedded option. Suppose the trade is settled at $t \geq 0$, becomes effective at $T_0 > t$ and matures at $T_N > T_0$. Define forward continuous survival probability as:

$$G(t,T) = \mathbb{E}\left(\exp\left(-\int_t^T r(s) + \sum_{\tau \in U(j) \cup U(i)} \lambda_\tau(s)ds\right) \mid \mathcal{F}_t\right) \tag{30}$$

and

$$G(t,T)\gamma_i(t,T) = \mathbb{E}\left(\sum_{\tau\in U(i)\setminus U(j)} \lambda_\tau(T)\exp\left(-\int_t^T r(s) + \sum_{\tau\in U(j)\cup U(i)} \lambda_\tau(s)ds\right) \mid \mathcal{F}_t\right) \quad (31)$$

Then, by the same logic as in Propositions 2 and 3:

$$\mathbb{E}\left(D(T)\mathbb{1}_{\{X_i>T\}}\mathbb{1}_{\{X_j>T\}} \mid \mathcal{F}_t\right) = G(t,T)D(t)\mathbb{1}_{\{X_i>t\}}\mathbb{1}_{\{X_j>t\}} \quad (32)$$

and

$$\mathbb{E}\left(\int_t^T D(s)\mathbb{1}_{\{X_j>s\}}d\mathbb{1}_{\{X_i>s\}} \mid \mathcal{F}_t\right) = D(t)\mathbb{1}_{\{X_i>t\}}\mathbb{1}_{\{X_j>t\}}\int_t^T G(t,s)\gamma_i(t,s)ds. \quad (33)$$

Let $T_1, ..., T_N = T$ be the swap premium payment dates and δ_n be the day count fraction for the interval $[T_{n-1}, T_n]$. Then, using the above equations, the fair premium of a forward-starting CDS over $[T_0, T_N]$, evaluated at $t < T_0$ is given by:

$$p(t) = \frac{LGD\int_t^{T_N} G(t,s)\gamma_i(t,s)ds}{\sum_{n=1}^N G(t,T_n)\delta_n}. \quad (34)$$

The call option price C with strike price K under the spot martingale measure (i.e. with the rolled savings account $B(t)$ as the numeraire) is given by:

$$\begin{aligned} C(0) &= \mathbb{E}\left(\sum_{n=1}^N e^{-\int_0^{T_n} r(s)ds}\mathbb{1}_{\{X_i>T_n\}}\mathbb{1}_{\{X_j>T_n\}}\delta_n \max(p(T_0)-K,0)\right) \\ &= \mathbb{E}\left(e^{-\int_0^{T_0} r(s)ds}\max(p(T_0)-K,0)\mathbb{1}_{\{X_i>T_0\}}\mathbb{1}_{\{X_j>T_0\}}\sum_{n=1}^N G(0,T_n)\delta_n\right). \end{aligned} \quad (35)$$

To handle (35) we shall introduce a new price system. Schönbucher (2004) suggests to use the fee stream for pricing CDS options. This may not be immediately clear, since the net present value of the premium payments can be zero, as both the reference credit and the counterparty can default before expiry. Luckily, we know that in such cases the option would be knocked out and the payoff would be zero as well, which indeed allows us to use $\mathbb{1}_{\{X_i>T_0\}}\mathbb{1}_{\{X_j>T_0\}}\sum_{n=1}^N G(0,T_n)\delta_n$ as the numeraire. We can then change the spot martingale measure to the forward survival measure $\mathbb{Q}$ and (35) becomes:

$$C(t) = \mathbb{E}^{\mathbb{Q}}\left(\max(p(T_0)-K,0) \mid \mathcal{F}_t\right)\mathbb{1}_{\{X_i>t\}}\mathbb{1}_{\{X_j>t\}}\sum_{n=1}^N G(t,T_n)\delta_n. \quad (36)$$

Hence, the change of numeraire allows to remove the premium stream from the expectation, leaving all the uncertainty in the distribution of the premium in the $\mathbb{Q}$ measure. As described in Schönbucher (2004), there is a whole range of models for the martingale dynamics of the swap premium. In what follows, we shall apply a version of the Libor market model. Consider a discrete time approximation of the protection leg:

$$\int_t^{T_N} G(t,s)\gamma_i(t,s)ds \approx \sum_{n=1}^N G(t,T_n)\delta_n L_n(t), \quad (37)$$

where $L_n(t)$ is the forward credit spread over $[T_{n-1}, T_n]$. Assume furthermore lognormal dynamics, i.e.

$$dL_n(t) = \ldots dt + L_n(t)\sigma_n(t)dW(t), \tag{38}$$

where $\sigma_n(t)$ is the instantaneous volatility and $W(t)$ a Wiener process (we omit the drift term for convenience). The swap premium now becomes a weighted average of Libor forward rates:

$$p(t) \approx \sum_{n=1}^{N} w_n(t)L_n(t), \tag{39}$$

3

with $w_n(t) = \frac{G(t,T_n)\delta_n}{\sum_n G(t,T_n)\delta_n}$. Hence, using Itô's lemma, the dynamics of the swap premium (34) can be approximated by:

$$\begin{aligned} dp(t) &\approx \ldots dt + \sum_{n=1}^{N} \frac{\partial p(t)}{\partial L_n(t)} L_n(t)\sigma_n(t)dW(t) = \ldots dt + \sum_{n=1}^{N} w_n(t)L_n(t)\sigma_n(t)dW(t) = \\ &= \ldots dt + p(t)\frac{\sum_{n=1}^{N} G(t,T_n)\delta_n L_n(t)\sigma_n(t)dW(t)}{\sum_{n=1}^{N} G(t,T_n)\delta_n L_n(t)} \end{aligned} \tag{40}$$

If forward curve movements are predominantly parallel (as assumed e.g. by Andersen and Andreasen, 2000, Gatarek, 2000 or Schönbucher, 2000 in a similar context), then (40) can be approximated further as:

$$dp(t) \approx \ldots dt + p(t)\frac{\sum_{n=1}^{N} G(0,T_n)\delta_n L_n(0)\sigma_n(t)dW(t)}{\sum_{n=1}^{N} G(0,T_n)\delta_n L_n(0)} = \ldots dt + p(t)\sigma_p(t)dW(t), \tag{41}$$

where $\sigma_p(t) = \frac{\sum_n G(0,T_n)\delta_n L_n(0)\sigma_n(t)}{\sum_{n=1}^{N} G(0,T_n)\delta_n L_n(0)} = \sum_n v_n\sigma_n(t)$ is the weighted average of forward rate volatilities. On the other hand, given that the swap premium is a relative price in the forward survival measure $\mathbb{Q}$, it must be a martingale:

$$dp(t) \approx p(t)\sigma_p(t)dW^{\mathbb{Q}}(t), \tag{42}$$

where $W^{\mathbb{Q}}$ is a Wiener process under the measure $\mathbb{Q}$. Hence, (36) can be solved to give the familiar Black (1976) option price formula for a call:

$$C(0) = [p(0)N(d_+) - KN(d_-)] \sum_{n=1}^{N} G(0,T_n)\delta_n, \tag{43}$$

where

$$d_{\pm} = \frac{\ln\left(\frac{p(0)}{K}\right) \pm \frac{1}{2}\Sigma(T_0)}{\sqrt{\Sigma(T_0)}} \tag{44}$$

Table 1: Term structure of credit spread volatility

Term	Credit spread volatility ($\sigma_n(t)$)
1×2	50%
2×3	45%
3×4	40%
4×5	35%
5×6	30%

and $\Sigma(T) = \int_0^T \sigma_p^2(t)dt$. An analogous formula for a put (receiver swaption) can be derived by put-call parity. Note that the dependence of the option price on counterparty credit risk is determined by the forward CDS premium, which impacts the moneyness of the option, as well as $\Sigma(T)$.

Example 3. Consider the economy from Example 2, but suppose now the protection sold by A is in the form of a swaption that expires in one year and the CDS extends over the next 5 years. Assume a flat risk free rate $r = 2\%$ and a flat term structure of credit spreads $L_n(0)$, as given by the hazard rates λ_A and λ_B, with volatilities given in Table 1. By (39), the CDS premium is (assuming for simplicity annual premium payments):

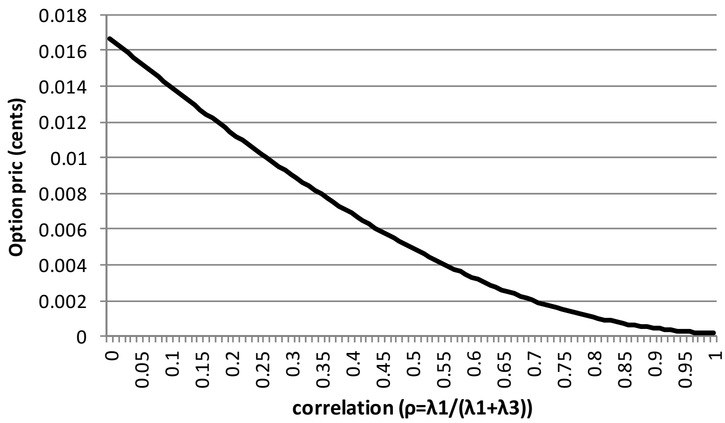
$$p(0) \approx \sum_{n=1}^{5} w_n(0)L_n(0) = 0.6(\lambda_B - \rho\lambda_A) \tag{45}$$

Spread volatility is given by:

$$\sigma_p = \sum_{n=1}^{5} \frac{\exp(-(r+\lambda_A+\lambda_B)n)\sigma_n}{\sum_n \exp(-(r+\lambda_A+\lambda_B)n)} \tag{46}$$

Using the data in Table 1, and assuming $\rho = 0.5$, we find $p(0) = 0.006$ and $\sigma_p = 40\%$. Figure 3 shows the value of the swaption with strike price equal to 60 bp as a function of default correlation between A and B. As expected, the value of the option is a decreasing function of correlation and the contract becomes worthless in case the defaults are co-monotonic.

Figure 3: Default swaption price as a function of default correlation (1×5 contract with $K = 60$ bps)



4 Conclusions

This paper proposes a new simple and intuitive model of dependent defaults that offers a continuous interpolation between independent and co monotonic defaults, thus allowing much greater flexibility than standard reduced form models where default correlation is induced by the correlation of default intensities and default times themselves are conditionally independent. The natural application of the model is in valuing products whose values depend on credit correlation between the counterparty and the reference entity. Thus, the prices of plain vanilla credit default swaps, first-to-default swaps and default swaptions in the presence of counterparty risk (and stochastic recovery rates) are derived as particular examples.

Several other applications and extensions of our framework can be suggested. First, the formulas presented above for the expected values of cash-flows under counterparty risk can be modified to include a (possibly stochastic) recovery rate on the counterparty, as opposed to merely the reference asset. Second, and perhaps more importantly, the dependence model could be calibrated to market data. Third, the framework could be applied outside of the scope of counterparty risk to model dependent defaults in basket credit derivatives or concentration risk in the context of recent financial stability policies (e.g. the creation of central clearing counterparties).

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