Narodowy Bank Polski

NBP Working Paper No. 241

Fan chart – a tool for NBP's monetary policy making

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Abstract

In this note we describe, in substantial detail, the methodology behind the construction of NBP's fan chart. This note is meant to help the readers interpret information contained in the mid-term projection, understand the differences in the predicted uncertainty between the projection rounds, and the usefulness of the projection for the monetary policy conduct. We describe the process which leads to the final projection, the methodology of estimation of the variance of the final forecast probability distribution, the method used for quantifying asymmetry of the fan chart and the role the two-piece normal distribution plays in it. Finally, we describe the analysis of the changes in the fan charts between the projection rounds and explain how the narrative associated with the projection is consistent with its assessment of risk.

Key words: Projection, balance of risk, variance, probability distribution, fan chart.

JEL: E31, E37, E59

Introduction

Narodowy Bank Polski (NBP) is, by its mandate, an inflation-targeting central bank. However, monetary policy decisions rest on the shoulders of the Monetary Policy Council (MPC) members, who make up an independent body of experts.[1] Hence, the NBP staff role is to support the MPC with expertise without influencing the MPC's decisions. The Economic Institute of the NBP draws up its macroeconomic projection three times a year, under the assumption of a constant interest rate path. The projection serves as background material for the Monetary Policy Council and is published as a part of the "Inflation Report". It does not reflect the MPC's view of future developments in the economy. Rather, it is the NBP's best qualified outlook of the Polish economy three years ahead, conveyed to the public with the purpose of supporting the MPC's decision making. The MPC has no obligation to take the presented forecast into consideration. However, an interest rate targeting monetary policy is, by its nature, a forward looking exercise and therefore a possibly precise understanding of future developments is crucial for the decision makers.

The main purpose of monetary policy is to keep inflation close to the target of 2.5% y-o-y, considered to be the long term equilibrium inflation rate for the Polish economy. A well-managed monetary policy which helps to avoid profound fluctuations is implemented ahead of expected phases of the business cycle. This is possible only if one takes into account the forecasted future economic developments, as well as the expected response of the economic variables to changes of the policy. Hence, a reliable and comprehensive macroeconomic projection has become one of the most important tools for monetary policy making. However, the projection carries uncertainty, which stems from imperfect forecasting tools, such as structural models involving only a simplified description of the economy. The uncertainty also stems from the limited reliability of available data and from the uncertainty about the predicted future scenario, i.e. the

[1] The only MPC member who works for the central bank is the NBP's President.

exogenous variables used in the forecasting exercise. On the other hand, the main forecasted variables: GDP and inflation, are inherently random variables and an accurate description of their expected behaviour can only be achieved through assessment of their probability distributions.[2]

For a long time central banks have published their macroeconomic forecasts comprising only the path of the main scenario. Although this method provided reference information about future developments in the economy, it lacked information about uncertainty and risk associated with the presented path. In 1996, the Bank of England presented an inflation forecast using a fan chart, a graph representing confidence intervals around the central path estimated based on two-piece normal distribution, see (Britton et. al., 1998).[3] Not only is such presentation much more faithful to the true nature of the forecast, which is in fact a future probability distribution of a random variable, but it also conveys essential knowledge about the uncertainty associated with the central scenario and allows us to track changes in the uncertainty between the forecasting rounds. Hence, most central banks have started presenting their forecasts using fan charts, and the technology used to produce fan charts has been steadily improved since then.

Future uncertainty associated with forecasted random variables depends heavily on the realisation of possible scenarios. At a certain point in time some scenarios are more probable than others, so upside or downside risks could dominate. This inevitably generates asymmetric uncertainty for the forecasted variables and it is not possible to represent this asymmetry with the normal distribution. This is why an asymmetric fan chart was developed. In central banking practice, it is constructed based on two-piece normal distribution, using the central scenario path as the mode of this distribution. The two sides of the distribution are allowed to have a different variance, and hence skewness is introduced to the graphical representation of the forecast, Britton et. al., (1998), Blix

[2] One should not confuse the variance of the projection error represented in the fan chart with the variance of the random variable.

[3] Similar procedure was introduced by the Sveriges Riksbank, see (Blix et. al., 1998).

et. al., (1998) and Wallis (1999).[4] Table 1 reports a summary of the usage of fan charts by several central banks around the world.[5]

This paper describes, in substantial detail, the methodology behind the construction of NBP's fan chart. It is meant to help the readers interpret information conveyed through the mid-term projection and understand differences in the perceived uncertainty between projection rounds.

[4] Nowadays the asymmetric fan chart became a popular tool for communication of the central bank forecast, see e.g. (Hucek et. al., 2009), and the methodology is continuously improved, (Elekdag et. al., 2009).

[5] Table 1 should viewed with caution since the central banks change and improve their methodologies quite frequently.

Central Bank:	Central path	Symmetry	Variance estimation	Asymmetry assessment	Expert judgment
Australia	Does not publish fan-charts	-	-	Narrative description of risk factors	-
Brazil	Mode	Asymmetric	-	-	-
Canada	Expected value	Asymmetric	Based on historical errors and expert judgment	Based on historical errors and expert judgment	Included in variance and asymmetry assessment
Chile	Mode	Asymmetric	Based on historical errors with expert judgment	Expert judgment	Included in variance and asymmetry assessment
Colombia	Mode	Asymmetric	Based on historical errors corrected for current uncertainty assessment	Expert judgment on uncertainty of endo- and exogenous variables	Included in asymmetry assessment
Czech Republic	Mode	Symmetric	Based on historical errors	-	Not included
European Central Bank	Does not publish fan-charts	-	-	-	-
Hungary	Mode	Asymmetric	-	-	-
New Zealand	Does not publish fan-charts	-	-	-	-
Norway	Expected value	Asymmetric	Combined from different models	Combined from different models	Included in variance and asymmetry assessment
Poland	Mode	Asymmetric	Based on historical errors and variance of conditioning variables	Iterative procedure based on alternative scenarios	Included in variance and asymmetry assessment
Slovakia	Mode	Asymmetric	Based on historical errors	-	Not included
Sweden	Mode	Symmetric	Based on historical errors	-	-
Switzerland	Does not publish fan-charts	-	-	-	-
United Kingdom	Mode	Asymmetric	Based on historical errors and variance of conditioning variables	Iterative procedure based on alternative scenarios	Included in variance and asymmetry assessment

Table 1: Fan charts in inflation targeting central banks.

The process

NBP's mid-term projection is prepared with the help of an econometric structural model of the Polish economy – the NECMOD, see (Budnik et. al., 2009a and 2009b). The model is constantly improved, contains all important macroeconomic aggregates that affect inflation processes and is believed to be a reliable tool representing the nuances of the Polish economy. The projection exercise, run three times a year, starts with a preparation of assumptions regarding the external environment of the Polish economy. Every four months a group of NBP's experts submit updated projections of these exogenous variables, which include, among others: food and oil prices, interest rates in the USA and the euro area, GDP growth and inflation abroad, etc. The experts provide their forecasts - estimates of uncertainty and the most likely scenarios, with qualitative assessment of the economic developments in the world markets which later serve as a basis for a consistent narrative of the mid-term projection. On the one hand, the central forecasted paths of the exogenous variables constitute the central scenario of the projection, whereas the submitted confidence intervals surrounding them play crucial role in estimation of the fan chart's width and its changes between projection rounds. For some variables such as interest rates abroad, exchange rates and commodity energy prices central path and uncertainty are derived directly from financial markets (e.g. option or futures markets), which allows for incorporating expectations of markets participations into the construction of the fan chart.

Once the assumptions for the central projection are collected the NECMOD model delivers the main scenario of the projection. There is a series of meetings among the Economic Institute experts where the necessary adjustments are discussed. Then, the forecasting team maps those corrections on the projection. After several iterations, the first draft of the forecast is ready. Since the process takes roughly a month, the so-called second cut-off for the data and exogenous assumptions has been introduced. Once the assumptions are updated, the projection is run once again. The final meeting of the forecasting team is held to discuss the consistency of the data, the projection and the narrative. At that point, alternative scenarios are prepared, too. The IE experts prepare several different sets of assumptions meant to describe different possible developments of the economic environment during the projection horizon. These scenarios provide an excellent mean of quantification of perceived risks associated with the projection, and, once assigned with probabilities of their realisation, determine the balance of risk illustrated on the fan chart.

Since the described forecasting technology is heavily based on the use of an econometric structural model, the bulk of the uncertainty of the final projection stems from the imperfectness of the forecasting tools and the quality of the data available. However, it is important to stress the fact that the projection and the assessment of risks associated with it are carried out with the assumption of no monetary policy change. Therefore, the fan chart does not convey any information about the uncertainty of future NBP interest rate level. On the one hand, a comparison of such projection to one where the market expectations about future interest rates are factored in is difficult. On the other hand, comparison of NBP's fan charts between the projection rounds is straightforward and conveys information about perceived change of riskiness in the environment surrounding the Polish economy.

The choice of probability distribution

The forecasting team's task is to prepare a tool for the monetary policy makers which will quantify the most likely future developments in the Polish economy, quantify the risk associated with the projection and provide a consistent narrative of the economic outlook. The fan chart provides information on the baseline scenario (central path), on the uncertainty surrounding the projection (bands of different shades of colours around the central view) and on the balance of risk. In general, in order to derive the fan chart one needs to estimate the probability distribution at each period of the forecast (formally the fan chart represents marginal distribution at each period of forecast). One of the ways to obtain fan chart would be running a great number of different possible scenarios using the forecasting model and obtain a distribution of the projection paths. If the risk was evenly distributed around the central projection, then a symmetric (e.g. normal) distribution would naturally occur for each period of the forecast. If the balance of risk was uneven, some asymmetric distribution would result.

Another way to obtain the fan chart is to assume that the probability distribution of the forecasted variables at each point is approximated by the probability density function that has a known analytical form, in practice by normal or two-piece normal distribution (TPN). The choice of the latter distribution is straightforward: it accommodates asymmetry and heteroscedasticity and is easy to compute, and in the case of symmetrical distribution of risk becomes a normal distribution.[6]

The density of the TPN distribution is based on two normal distributions with the same central value, but different variances, and is fully specified with three parameters: the mode μ – the most likely realisation of the random variable and two measures of standard deviation - σ_1, σ_2. In other words, it is proportional to the normal distribution with mean μ and standard deviation σ_1 to the left of the mode, whereas to the right of the mode to the normal distribution with mean μ and

[6] For more information about the TPN distribution, see (John, 1982) and (Johnsons et. al., 1994).

standard deviation σ_2, see Figure 1. To be able to fully specify the TPN distribution in practise the forecasting team needs central path of the projection, assessment of uncertainty and a measure of asymmetry. All these components are described in detail in following sections.

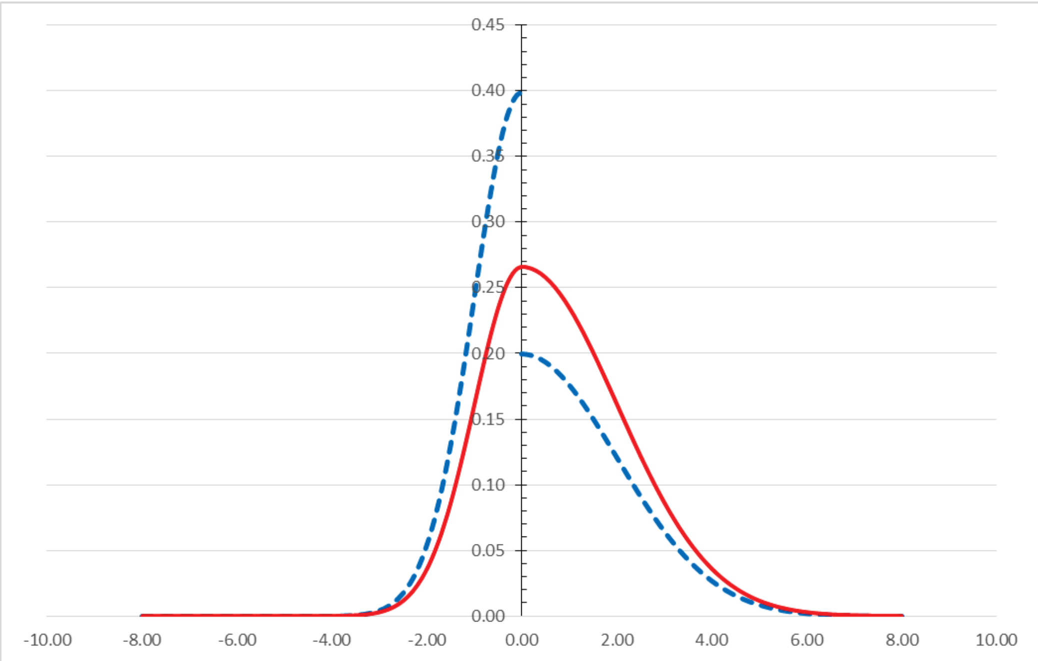

Figure 1: The probability density function of the two-piece-normal distribution. Dashed line: left half of $N(\mu, \sigma_1)$ and right half of $N(\mu, \sigma_2)$ distributions with $\mu = 0$ and $\sigma_2 > \sigma_1$. Solid line: combined two-piece normal distribution.

When $\sigma_1 = \sigma_2$, the TPN distribution transforms into a normal distribution with standard deviation $\sigma = \sigma_1 = \sigma_2$, and the mean, the mode and the median equal μ. When $\sigma_2 > \sigma_1$, we will say that the upper risk dominates, hence the mean > median > mode. The opposite is true when the risk of lower than the mode values dominates. Figure 2 compares right-skewed two-piece normal distribution where the mean is bigger than the mode, and a great probability mass is placed to the right of the mode and the symmetric case where the standard deviations of both sides of the mode are equal.

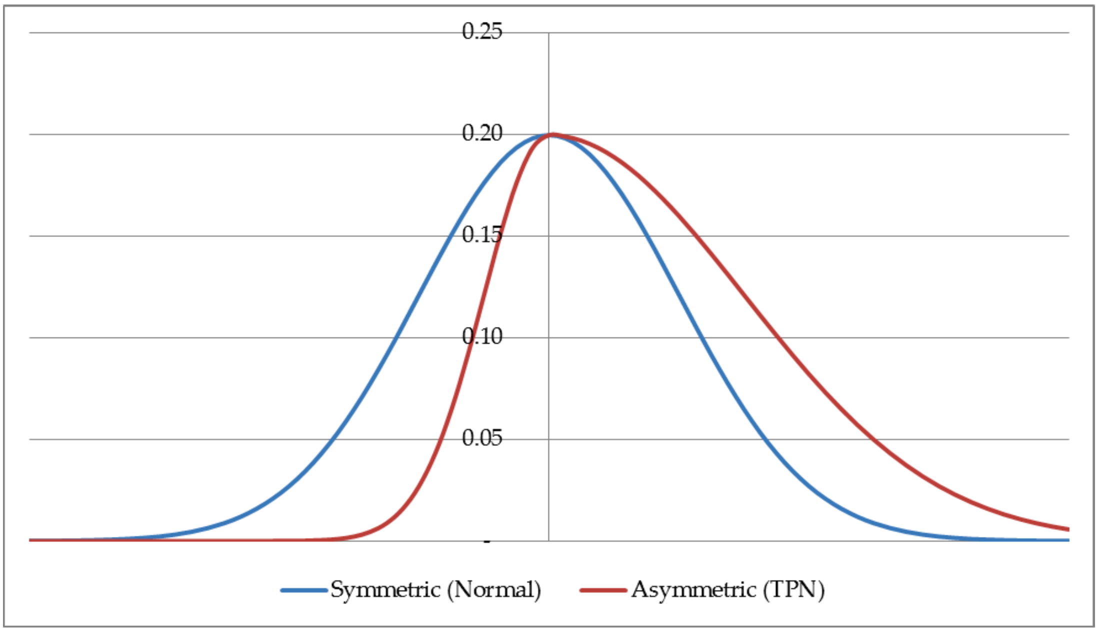

Figure 2: The probability density function of the two-piece-normal distribution. Blue line: $\sigma = \sigma_2 = \sigma_1$*. Red line* $\sigma_2 > \sigma_1$.

The central path of projection

The projection exercise is a deterministic solution of the structural econometric models given a set of assumptions about exogenous variables. Since the paths of the conditioning variables are interpreted as their most likely future outcomes, the solution from the model, given the theoretical structure of the economy, is understood as the most likely future outcome of the forecasted variables, e.g.: GDP and inflation. However, one should bear in mind that from the theoretical point of view, presentation of a projection by means of a single path of the most likely values does not make much sense, since the probability of any single value of a continuous random variable is 0. This is why the fan chart is essential if an accurate representation of forecast of a random variable is required.

The baseline scenario of the forecast is determined by the extreme values of possible realisations of the conditioning random variables. Hence, it is a natural candidate for the most likely value of the split distribution – the **mode**.[7] However, the clear advantage of the split value of the TPN in terms of the interpretation is paralleled by the fact that the mode does not use all the available information as the expected value, the **mean**, does. If the risk is evenly distributed around the mode, the distinction disappears as the probability distribution becomes symmetric and the mode is equal to the mean. If the mode is different from the mean, then the probability mass on one side of the mode will be different than on the other. This way we obtain a convenient model of the balance of risks and quantification of the probability of realisation of the forecasted variable below and above of the central path. Now, the future developments of the forecasted variables can be interpreted in terms of deviations from the central path.

[7] Table 1, shows that the central scenario of the forecast indeed is usually used as the mode of the fan chart.

Prediction uncertainty

The uncertainty associated with the prediction describes the probability that the actual realisation of the endogenous variables will differ from the forecasted central path. It is important to stress that it is a forward looking assessment of risk and should not be simply extrapolated based on historically made errors in the previous projection rounds. The estimated uncertainty has to be affected by the assessment of risk associated with future developments, i.e. the risk associated with the projections of conditioning variables. The degree of uncertainty, which is the spread of the fan chart is usually measured by the variance of the probability distribution.

The methodology employed to estimate the final fan chart variance is based on the procedure described by Blix and Sellin (1998) and Pinheiro and Esteves (2010). We assume a linear relationship between the variance and covariances of the distributions of the exogenous variables and the variance of the final fan chart. This assumption allows us to combine the uncertainties associated with the conditioning variables, weighted by their impact on the endogenous variable, as specified in the forecasting model. However, the variance of the forecasted variable's fan chart is not simply a linear function of the conditioning variables uncertainty, but also a function of historically made forecasting errors which stem from the imperfect structure of the model, the estimated parameters of the equations, etc. Therefore, we approach the estimation of variance by assuming a reference level of uncertainty of the final projection. A natural candidate is the variance of the historical projection errors by which we mean both the forecast errors and the errors made while forecasting the conditioning variables.

Once the historical variance is computed, the fan chart variance is updated with uncertainty introduced by the risk factors in the current projection round. The weight of the particular risk factor on the forecasted variable is easily obtainable from the forecasting model as an impulse response. The current spread of the exogenous variable's distribution is compared to its past average standard deviation. The result is called the **scaling factor** and, if for example it is equal to 1.2 it means that the current standard deviation is by 20% bigger than the historical average. Hence, the risk factor will be responsible for widening the final fan chart proportionally to its impulse response for each projection period. In other words, the scaling factors serve as multipliers for the current exogenous uncertainty measure for each conditioning variable. The process behind variance estimation is explained in detail in the Appendix D.[8]

[8] There is an important difference between the inflation and the GPD fan charts. The GDP fan chart displays, apart from the projection periods (at most 11 in the March round), seven periods of historical uncertainty. This spread is computed as a means square error (MSE), an unbiased estimator of variance, and reflects revisions of the national accounts data.

Prediction asymmetry

The conditioning variables carry over asymmetry into the final spread of the fan chart.[9] This asymmetry is reflected in the asymmetry of the forecasted variables only to the degree the endogenous variable responds to the shocks in the conditioning variable. Hence, usually the balance of risk is not conveyed in a satisfactory manner into the final fan chart. Also, a pure measure of asymmetry of a conditioning variable's distribution does not carry a satisfactory narrative of its future possible developments and the resulting overall outlook. This is why a set of alternative scenarios is run during every projection round. Those scenarios are designed to resemble an economically consistent picture of possible future states of the economic environment. All scenarios, including the baseline projection, have assigned probabilities of realization based on expert judgement. A weighted average of all the forecasted paths for each quarter of the projection horizon produces expected values of the final distribution of the endogenous variables. Hence, if the experts believe there is a high probability of less optimistic developments abroad, the expected value of the distributions is likely to be lower than the central path, and the probability of realisation of the projected variable, e.g. inflation, below the central path will be higher than 50%.

It is easy to see that, apart from very rare cases when the exogenous uncertainty is symmetric and the alternative scenarios cancel each other out, the most likely scenario for the endogenous variable does not coincide with the expected value. However, once the mode and the expected value for the final TPN distribution are known it is possible to obtain the variances for both sides of the TPN distribution as was shown in the appendix B.

[9] A good example of such asymmetric uncertainty is the euro short term interest rate in the November 2014 projection round when the ECB's interest rates were already so low that the further easing of monetary policy meant negative interest rates. Hence, the standard deviation of the forecast was much bigger to the north of the point forecast.

Probability intervals

In order to derive graphical representation of uncertainty, i.e. fan chart, one needs to supply confidence intervals of specified coverage, that is intervals in which future values of macroeconomic variable will lie with a given amount of probability. Generally, there are a number of ways to derive probability intervals with specified coverage (there are in fact infinite numbers of them), for example in the case of standard normal distribution both intervals (−1.44, 1.96) and (−1.65, 1.65) give 90% of probability.

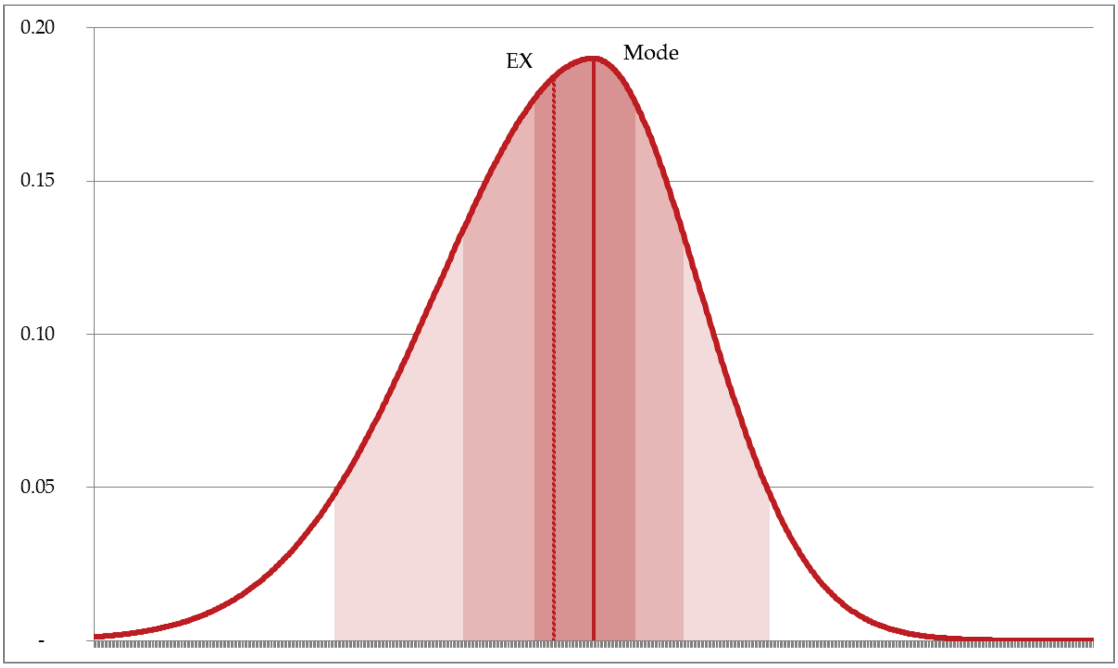

Figure 3: The probability density function of the two-piece normal distribution with $\sigma_1 > \sigma_2$, where 60% of the probability mass lies to the left from the mode.

One standard and common way to overcome this problem is to provide quantile-based probability intervals which are centered around the median and have equal-sized tails.[10] In other words, the $100(1-\beta)\%$ central interval is defined by the $\beta/2$ and $1-\beta/2$ quantiles, hence e.g. 90% of central interval of distribution is formed by the 5th and the 95th percentile. In the case of a symmetrical distribution, e.g. the normal distribution, it is centered at both the mode, the mean and the median, since as mentioned earlier, these measures coincide. The situation changes

[10] In other words interval (a, b) such that $a < mode < b$ and $\Pr(x < a) = \Pr(x > b)$.

significantly in the case of a highly skewed (asymmetric) distribution, where the central bands are not be symmetric anymore. In some extreme cases it is possible to show the mode of the distribution lying outside some central bands, which could be highly misleading. Figure 3 illustrates TPN distribution skewed to the left with 30%, 60% and 90% probability intervals.

The problem can be resolved by using the so called "highest probability density" (HPD) intervals, (Casella and Berger, 2002).[11] The HPD intervals are centered around the mode, hence the most likely value of the forecast in the case of the fan chart, by construction, always lies within a specified interval. The value of the density function for every point inside HPD interval is greater than that of any point outside the interval. The HPD intervals are especially useful when a random variable has a unimodal distribution, like the TPN distribution.[12] In such case, HPD interval is shorter than any other probability interval with the same coverage. Since, the HPD interval does not necessarily produce the equal-tails outside interval, it is particularly useful in displaying asymmetric uncertainty of a forecast. The derivation of HPD intervals for the two-piece normal distribution, which has unimodal and continuous probability density function, is relatively easy and facilitates computational burden considerably, see Appendix B.[13]

[11] Some authors use term highest density interval (HDI) or, in case of multivariate distributions, highest probability region (HDR).

[12] In case of multimodal distribution HPD interval can be sum of disjoint intervals.

[13] Julio (2006), gives a detailed explanation of this method.

Yearly data

Sometimes it is useful to report estimated uncertainty of a forecast in annual terms. Then, the fan chart will consist of a series of distributions for each year instead of a quarter. However, the fan chart by its construction, as presented above, does not reflect the dependencies between different time periods (quarters), which will also be true for uncertainty for annual values constructed in a similar way. Therefore, for yearly data the following procedure is derived. Firstly, for each quarter we have the TPN distribution, although the joint distribution for all quarters is not specified. Secondly, using errors from past forecasting rounds, their correlation matrix is computed. Finally, using the above-mention distributions for each quarter and correlation matrix we simulate quarterly growth rates in a manner similar to that described by Lurie et. al. (1998).

Once we obtain a sample of correlated quarterly dynamics, which are transformed to the respective annual dynamics, the simulated values are then approximated by two-piece normal distribution, because the sum of TPN random variables is not a TPN random variable anymore. Since the mode of this distribution is known (central path), a maximum likelihood estimation method is then employed to derive the remaining parameters (σ_1, σ_2). With all three parameters of the distribution for each year the desired HPD intervals are computed.

The fan chart procedure

As mentioned before, the TPN distribution is fully specified by a triple $(\mu, \sigma_1, \sigma_2)$, so all we need is to determine these parameters for every period of the projection i.e. for each quarter/year to be considered. The main steps of the procedure are summarized and listed below.

Quarterly data

For each horizon $h = 1, \dots, H$, where H is maximum allowed horizon, we obtain:

A. Mode

- Computation of the central scenario for each of the endogenous variables – the most likely paths: μ_h.

B. Asymmetry

- Computation of the alternative scenarios of the future developments of the endogenous variables,
- Each scenario receives a probability of realization based on expert judgment. Then, the calculation of the expected value for forecasted variables is possible.
- The difference between expected value and the mode is an approximation of the asymmetry of the fan chart.

C. Variance

- Computation of the smoothed forecast errors for the endogenous variables. (vector $\boldsymbol{e}$)
- Determination of the impulse responses of all endogenous variables to the unitary shock in each of the exogenous variables, which remains unchanged constant between re-estimation rounds.[14] (matrix $\boldsymbol{C}$)

[14] The NECMOD model is re-estimated once a year, hence three consecutive forecasting rounds use the same set of impulse responses.

- Determination of the smoothed forecast errors of the exogenous variables and their correlation matrix, and therefore covariance matrix of these errors. (matrix $\boldsymbol{Z}$)
- Determination of the **pure forecasting error** for the endogenous variables which is the smoothed forecasting error minus the exogenous error. ($\boldsymbol{e} - diag(\boldsymbol{CZC}')$)[15]
- Determination of matrix of **scaling factors**. (matrix $\boldsymbol{S}$)
- Determination of the **corrected exogenous error** given by $diag(\boldsymbol{C} \cdot \boldsymbol{S} \cdot \boldsymbol{Z} \cdot \boldsymbol{S} \cdot \boldsymbol{C}')$.
- The final fan chart variance is a sum of the **pure forecasting error** and the **corrected exogenous error,** that is $(\boldsymbol{e} - diag(\boldsymbol{CZC}') + diag(\boldsymbol{C} \cdot \boldsymbol{S} \cdot \boldsymbol{Z} \cdot \boldsymbol{S} \cdot \boldsymbol{C}'))$.

D. Solve the system of equations involving mode, variance and skewness to determine the triple $(\mu_h, \sigma_{1h}, \sigma_{2h})$.

E. Confidence bands
- Determination of the α-th HDP intervals for given endogenous variable.
- Determination of the probability tables for given realizations of the forecasted variables.

Yearly data

For each horizon $h = 1, \dots, H$, where $H = 3$ is the maximum allowed horizon we:

- compute correlation matrix of past forecasting errors,
- simulate the quarterly growth rates,
- convert quarterly dynamics to respective yearly counterparts (mean transformation),
- calculate HPD confidence bounds.

[15] $diag(\boldsymbol{X})$ refers to main diagonal of matrix $\boldsymbol{X}$.

An illustration

For the sake of illustration of the fan chart procedure we present a detailed example.[16] Although fan charts are generated for many variables, e.g. GDP components for the need of short-term projections, here we will limit the vector of endogenous variables to two of those:

$$D = \begin{bmatrix} inflation \\ GDP \end{bmatrix}$$

The vector of the conditioning variables for the need of the fan charts is restricted to the variables whose impact on the forecasted variables, due to the structure of the model, is significantly different from zero. Hence, the vector of the exogenous variables takes the form:

$$X = \begin{bmatrix} foreign\ GDP \\ foreign\ potential\ output \\ foreign\ price\ deflator \\ oil\ price \\ gas\ price \\ coal\ price \\ food\ price \\ EUR/USD\ exchange\ rate \\ euro\ interest\ rate \\ US\ interest\ rate \end{bmatrix}$$

The direct and indirect impact of the exogenous variables on GDP and inflation is known as matrix of impulse response functions updated every year after re-estimation of the NECMOD model. This ensures a precise and updated measure of impact of the uncertainty associated with an exogenous variable on the forecasted variable's uncertainty. Table 2 shows the matrix of the impulse responses of the Consumer Price Index to shocks in conditioning variables.

[16] For the sake of this illustration we have crafted some new parameters and run a projection exercise similar to that from the November 2014 projection round. Therefore, the example is internally consistent, however, does not report the true November 2014 projection.

The uncertainty associated with exogenous variables in the current projection round is judged based on comparison to the historical contribution of the exogenous variables uncertainty to the variance of the fan chart. Hence, the next step of the fan chart procedure is the computation of historical errors of the exogenous predictions. Then, mean squared errors (MSE), the unbiased estimates of the variance of the forecast errors, for each quarter and each variable are computed. Since the number of observations for particular quarters of the projection differs, the errors are smoothed. At the same time, a covariance matrix of the historical errors of exogenous variables is estimated.

	pva_ext	gdp_ext	gdp_pot_ext	i_3m_eur	i_3m_us	s_usd_eur	p_food_base	p_oil	p_gas_gbp	p_coal_usd
h=1	0.012	0.020	-0.009	0.013	0.001	1.075	0.507	1.316	0.000	0.002
h=2	0.023	0.035	-0.059	0.032	0.002	1.000	1.470	0.873	-0.001	0.149
h=3	0.021	0.053	-0.084	0.036	0.003	0.855	0.729	0.418	0.002	0.312
h=4	0.023	0.066	-0.102	0.030	0.002	0.658	0.297	0.438	0.042	0.431
h=5	0.011	0.044	-0.106	0.009	0.002	-0.566	-0.409	-1.010	0.083	0.494
h=6	0.001	0.033	-0.060	-0.019	0.000	-0.587	-1.458	-0.695	0.116	0.307
h=7	0.004	0.018	-0.041	-0.030	-0.001	-0.566	-0.728	-0.404	0.132	0.071
h=8	0.004	0.008	-0.029	-0.031	-0.002	-0.458	-0.277	-0.509	0.101	-0.115
h=9	0.004	0.014	-0.024	-0.029	-0.003	-0.340	-0.061	-0.405	0.063	-0.223

Table 2: An example of impulse responses of the CPI to shocks in the exogenous variables. (Matrix C)

The resulting smoothed MSE constitute a matrix of standard deviations for the exogenous variables. Table 3 shows an example the MSE matrix.

	pva_ext	gdp_ext	gdp_pot_ext	i_3m_eur	i_3m_us	s_usd_eur	p_food_base	p_oil	p_gas_gbp	p_coal_usd
h=1	0.166	0.185	0.133	0.152	0.169	0.169	0.219	0.139	0.144	0.175
h=2	0.207	0.333	0.211	0.376	0.279	0.325	0.241	0.154	0.190	0.191
h=3	0.227	0.395	0.246	0.459	0.326	0.388	0.253	0.163	0.212	0.200
h=4	0.240	0.433	0.268	0.510	0.356	0.427	0.261	0.169	0.226	0.206
h=5	0.250	0.461	0.284	0.546	0.378	0.455	0.268	0.173	0.237	0.210
h=6	0.258	0.482	0.296	0.574	0.395	0.477	0.273	0.176	0.245	0.214
h=7	0.265	0.500	0.306	0.596	0.408	0.495	0.277	0.179	0.252	0.217
h=8	0.270	0.514	0.315	0.615	0.420	0.509	0.280	0.182	0.257	0.219
h=9	0.275	0.527	0.322	0.631	0.429	0.522	0.283	0.184	0.262	0.221

Table 3: Standard errors of the exogenous variables – historical levels.

10.1 Scaling Factors

All the exogenous variables' projections come with 95% confidence intervals which will be incorporated into final variance of the fan chart. In most of the cases the intervals are asymmetric, resembling an unbalanced risk associated with the variable. Hence, the uncertainty is fitted into a TPN distribution. At the same time, normalized current uncertainty (standard deviation) associated with every exogenous variable is compared to the weighted average of the previous forecasts' spreads. This way a vector of Scaling Factors (SF) is estimated for each exogenous variable. An example of the resulting matrix is displayed in Table 3.

	pva_ext	gdp_ext	gdp_pot_ext	i_3m_eur	i_3m_us	s_usd_eur	p_food	p_oil	p_gas_gbp	p_coal_usd
h=1	1.02	0.50	0.98	1.10	0.92	1.04	0.93	1.20	1.11	1.06
h=2	0.98	0.50	1.08	1.01	1.00	1.05	0.93	1.20	0.92	0.97
h=3	0.86	0.50	0.94	0.99	0.91	0.88	0.88	1.20	0.85	0.93
h=4	1.02	0.50	0.94	1.05	1.18	1.07	1.03	1.20	1.01	1.11
h=5	1.07	0.50	1.08	0.97	1.11	1.02	0.96	1.20	0.96	1.16
h=6	0.94	0.50	1.21	0.83	0.96	0.93	0.99	1.20	1.10	1.07
h=7	1.02	0.50	1.17	1.04	1.00	1.03	1.00	1.20	1.03	0.97
h=8	0.91	0.50	1.10	0.81	0.99	0.80	0.99	1.20	1.17	0.92
h=9	1.01	0.50	0.93	1.02	0.92	0.95	0.92	1.20	0.80	0.90

Table 4: A matrix of scaling factors for the exogenous uncertainty associated with each conditioning variable. (Matrix S)

10.2 Ex-post forecasting errors of the endogenous variables

The history of the forecasting rounds provides information about the ex-post errors of the projection. The root means square error is computed for every projection horizon and smoothed since the number of observed errors differs for different forecasting periods. Table 5 shows an example of the matrix of the RMSE for both endogenous variables.

	inf	gdp
h=0	0.43	0.95
h=1	0.76	1.21
h=2	1.05	1.57
h=3	1.23	1.75
h=4	1.35	1.86
h=5	1.43	1.93
h=6	1.50	1.96
h=7	1.54	1.99
h=8	1.58	2.00
h=9	1.60	2.01

Table 5: Ex-post projection errors for endogenous variables.

10.3 Corrected exogenous errors

The uncertainty associated with the conditioning variables is responsible for changes in the spread of the final distribution. The resulting impact on the endogenous variables forecast is obtained by multiplying the covariance matrix of the exogenous variables by the corresponding matrices of impulse responses, Table 6.

	inf	gdp
h=1	0.00	0.03
h=2	0.01	0.17
h=3	0.03	0.34
h=4	0.08	0.54
h=5	0.15	0.66
h=6	0.21	0.83
h=7	0.24	0.95
h=8	0.33	1.06
h=9	0.38	1.20

Table 6: Exogenous errors.

However, as easily seen from the scaling factors matrix, the expected uncertainty in the surrounding environment is deemed to be smaller on average than its historical levels (with the exception of the gas and oil prices). Hence, the resulting corrected exogenous errors take the form displayed in Table 7.

	inf	gdp
h=1	0.01	0.02
h=2	0.06	0.03
h=3	0.02	0.19
h=4	0.03	0.38
h=5	0.10	0.46
h=6	0.21	0.43
h=7	0.16	0.56
h=8	0.16	0.69
h=9	0.20	0.77

Table 7: Corrected exogenous errors.

10.4 Pure forecasting error

The pure forecasting errors - the errors the forecasters would make with the assumption that the conditioning variables are predicted perfectly, are estimated as a difference between the forecasted errors and the estimated measure of exogenous uncertainty, Table 8.

	inf	gdp
h=1	0.18	0.88
h=2	0.57	1.28
h=3	1.07	2.12
h=4	1.43	2.54
h=5	1.67	2.81
h=6	1.85	2.88
h=7	2.00	2.91
h=8	2.05	2.89
h=9	2.11	2.81

Table 8: Pure forecasting errors.

10.5 The asymmetry of the fan chart.

The expert judgement plays a central role in determining the asymmetry of the prediction distribution. Once the alternative scenarios are specified and the central projection is conducted, the experts decide on probabilities of realisation of particular scenario. The weighted average of the scenarios gives the expected value of realisations of the forecasted random variable for each forecasting period. The distance between the mode (central path) and the expected value is a measure of asymmetry (skewness) of the fan chart, Table 9.[17] Figure 4 illustrates the spread of the scenarios' paths and the resulting expected value of the forecast.

	Central path	pesymistic scenario	optimistic scenario
Assigned probabilities:	0.55	0.4	0.05
h=1	-0.21	-0.27	-0.18
h=2	0.44	0.29	0.53
h=3	1.08	0.81	1.24
h=4	1.28	0.84	1.53
h=5	1.53	0.95	1.87
h=6	1.46	0.74	1.86
h=7	1.48	0.61	1.94
h=8	1.69	0.68	2.20
h=9	1.81	0.66	2.37

Table 9: Scenarios with assigned probabilities of realization based on expert judgment.

[17] In principle, it is possible that the distance would be zero. For instance, when the pessimistic scenario and the optimistic scenarios weighted by their probabilities cancel each other out. Then, the TPN distribution will become Normal distribution and the fan chart will display an even distribution of risk.

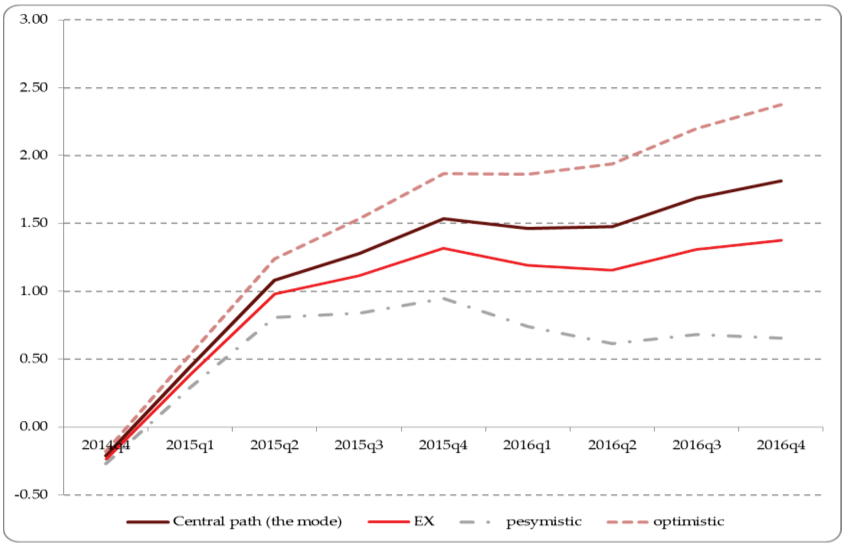

Figure 4: The three scenarios and the expected value of the forecast.

Once all the elements of the forecast are estimated, the parameters of two-piece normal distribution can be computed. With the parameters of the TPN distribution and the probability intervals associated with the forecast in place, the balance of risk can be quantified, Table 10. The HPD intervals for the need of the fan chart are computed in Table 11 and displayed in Figure 5 in the form of the final CPI forecast.

				TPN parameters			
	Mode	EX	EX-mode	Variance	sigma 1	sigma 2	Balance of risk*
h=1	-0.21	-0.23	-0.02	0.19	0.45	0.42	0.52
h=2	0.44	0.39	-0.06	0.63	0.83	0.76	0.52
h=3	1.09	0.98	-0.11	1.10	1.11	0.98	0.53
h=4	1.27	1.11	-0.16	1.46	1.30	1.11	0.54
h=5	1.54	1.32	-0.22	1.77	1.46	1.19	0.55
h=6	1.45	1.18	-0.27	2.05	1.60	1.26	0.56
h=7	1.48	1.15	-0.33	2.15	1.67	1.25	0.57
h=8	1.69	1.31	-0.37	2.22	1.71	1.25	0.58
h=9	1.81	1.38	-0.44	2.31	1.78	1.23	0.59

* The probability of realisation of the inflation path below the mode

Table 10: The parameters of the final TPN distribution and the balance of risk.

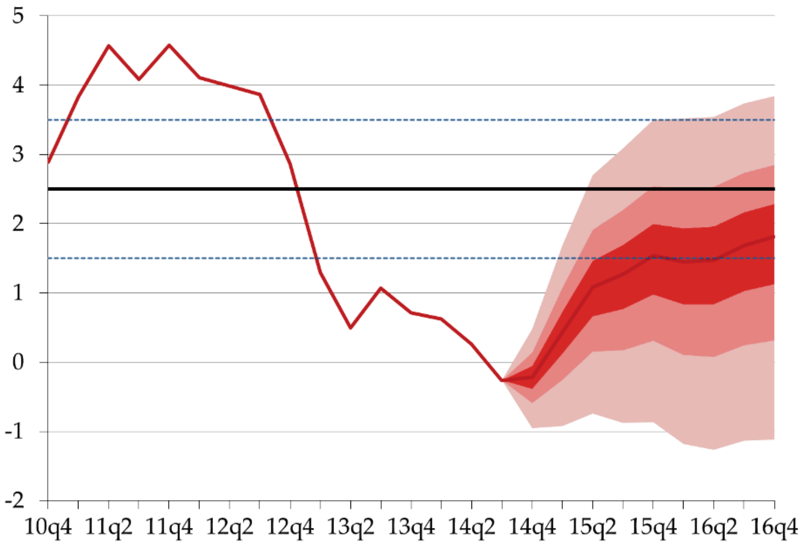

Figure 5: CPI inflation fan chart.

(%)	30%	50%	60%	90%
h=1	(-0,4;-0,1)	(-0,5;0,1)	(-0,6;0,1)	(-1,0;0,5)
h=2	(0,1;0,7)	(-0,1;1,0)	(-0,3;1,1)	(-0,9;1,7)
h=3	(0,7;1,5)	(0,3;1,8)	(0,2;1,9)	(-0,7;2,7)
h=4	(0,8;1,7)	(0,4;2,0)	(0,2;2,2)	(-0,9;3,1)
h=5	(1,0;2,0)	(0,6;2,3)	(0,3;2,5)	(-0,9;3,5)
h=6	(0,8;1,9)	(0,4;2,3)	(0,1;2,5)	(-1,2;3,5)
h=7	(0,8;2,0)	(0,4;2,3)	(0,1;2,5)	(-1,3;3,5)
h=8	(1,0;2,2)	(0,5;2,5)	(0,2;2,7)	(-1,1;3,7)
h=9	(1,1;2,3)	(0,6;2,6)	(0,3;2,9)	(-1,1;3,8)

Table 11: Confidence intervals for CPI inflation projection.

(%)	below 1,5%	below 2,5%	below 3,5%	below central path	between (1,5%; 3,5%)
h=1	1,00	1,00	1,00	0,52	0,00
h=2	0,92	1,00	1,00	0,52	0,08
h=3	0,68	0,93	0,99	0,53	0,31
h=4	0,62	0,88	0,98	0,54	0,36
h=5	0,54	0,81	0,96	0,55	0,42
h=6	0,57	0,82	0,95	0,56	0,38
h=7	0,58	0,82	0,95	0,57	0,38
h=8	0,53	0,78	0,94	0,58	0,41
h=9	0,51	0,76	0,93	0,59	0,42

Table 12: Probability of CPI inflation realization.

Fan chart decomposition

The decomposition of the fan chart is yet another important exercise conducted during each projection round. Since the methodology allows for an introduction of the uncertainty brought in by every conditioning variable, and the impact of these conditioning variables is tracked through the impulse response functions, it is possible to quantify the impact of the uncertainty of each of the exogenous variables on the variance of the final fan chart. This way we obtain a picture of the sources of changes in uncertainty of the forecast between projection rounds which allows for a consistent narrative of the risk surrounding the projection of the Polish economy. Table 13 represents the decomposition of the impact of exogenous uncertainty on the variance of the final CPI fan chart, and Figure 6 illustrates this impact.

	pva_ext	gdp_ext	gdp_pot_ext	i_3m_eur	i_3m_us	s_usd_eur	p_food_base	p_oil	p_gas_gbp	p_coal_usd	sum
h=1	0.000	0.000	0.000	0.000	0.000	0.000	0.003	0.004	0.000	0.000	0.008
h=2	0.000	0.000	0.000	0.000	0.000	0.000	0.031	0.000	0.000	0.000	0.032
h=3	0.000	0.000	0.000	0.000	0.000	0.000	0.006	-0.001	0.000	0.000	0.006
h=4	0.000	0.000	0.000	0.000	0.000	0.001	-0.001	0.001	0.000	0.000	0.002
h=5	0.000	0.000	0.000	0.000	0.000	0.002	0.009	0.017	0.000	0.001	0.029
h=6	0.000	0.000	0.000	0.000	0.000	0.003	0.056	0.025	0.000	0.000	0.084
h=7	0.000	0.000	0.000	0.000	0.000	0.001	0.031	0.030	0.000	0.001	0.064
h=8	0.000	0.000	0.001	0.000	0.000	0.000	0.018	0.037	0.000	0.004	0.059
h=9	0.000	0.001	0.001	0.000	0.000	0.000	0.011	0.037	0.000	0.005	0.055

Table 13: Decomposition of the change of variance of the CPI inflation fan chart.

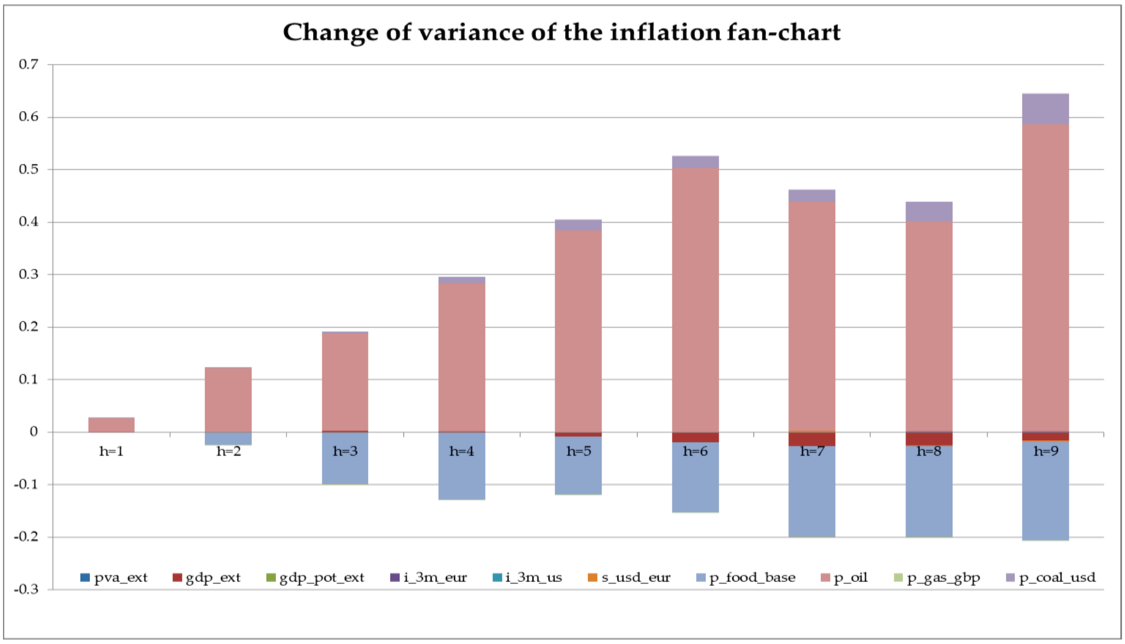

Figure 6: Decomposition of the impact of exogenous uncertainty on the variance of fan chart compared to previous projection round.

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Appendix A. The two-piece normal distribution and its properties

Random variable X is said to have a two-piece normal distribution (TPN) or a split-normal distribution (SN) with parameters μ, σ_1 and σ_2 if it has a probability density function (pdf) given by:[18]

$$f(x;\mu,\sigma_1,\sigma_2) = \begin{cases} C \cdot exp\left[-\frac{(x-\mu)^2}{2\sigma_1^2}\right], & if \quad x < \mu \\ C \cdot exp\left[-\frac{(x-\mu)^2}{2\sigma_2^2}\right], & if \quad x \geq \mu \end{cases}, for \;\; x \in R \qquad \text{(A1)}$$

where μ is the mode of the variable X and the parameters $\sigma_1 > 0$, $\sigma_2 > 0$ are the standard deviations of the respective "halves" of the distribution (Figure 1); $C = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sigma_1+\sigma_2}$ is normalizing constant such that (A1) integrates to unity. TPN density is completely specified by the triple $(\mu, \sigma_1, \sigma_2)$.

Probabilities can be computed using (A1) as follows:

$$P(L_1 \leq X \leq L_2) = \int_{L_1}^{L_2} f(x)dx =$$

$$= \begin{cases} \frac{2\sigma_1}{\sigma_1+\sigma_2}\left[\Phi\left(\frac{L_2-\mu}{\sigma_1}\right) - \Phi\left(\frac{L_1-\mu}{\sigma_1}\right)\right], & if \; L_1 < L_2 \leq \mu \\ \frac{2\sigma_2}{\sigma_1+\sigma_2}\left[\Phi\left(\frac{L_2-\mu}{\sigma_2}\right) - \Phi\left(\frac{L_1-\mu}{\sigma_2}\right)\right], & if \; \mu \leq L_1 < L_2 \\ \frac{2\sigma_2}{\sigma_1+\sigma_2}\left[\sigma_2\Phi\left(\frac{L_2-\mu}{\sigma_2}\right) - \sigma_1\Phi\left(\frac{L_1-\mu}{\sigma_2}\right) + \frac{\sigma_1-\sigma_2}{2}\right], & if \; L_1 \leq \mu < L_2 \end{cases}, \qquad \text{(A2)}$$

where Φ(x) is the cumulative distribution function of the standard normal distribution.

It is easy to verify that balance of risks is given by: $p = P(X \leq \mu) = \frac{\sigma_1}{\sigma_1+\sigma_2}$.

The mean value, the variance and the 3rd central moment of the TPN distribution are given by, respectively:

$$E[X] = \mu + \sqrt{\frac{2}{\pi}}(\sigma_2 - \sigma_1), \qquad \text{(A3)}$$

$$V[X] = \left(1 - \frac{2}{\pi}\right)(\sigma_2 - \sigma_1)^2 + \sigma_1\sigma_2, \qquad \text{(A4)}$$

$$T[X] = E[(X - E[X])^3] = \sqrt{\frac{2}{\pi}}(\sigma_2 - \sigma_1)\left[\left(\frac{4}{\pi} - 1\right)(\sigma_2 - \sigma_1)^2 + \sigma_1\sigma_2\right]. \qquad \text{(A5)}$$

[18] Another parameterization of TPN distribution may be found in Johnson et. al. (1994).

Since the 3rd moment of the TPN distribution is not easy to use, a different, proportional measure of skeweness is adopted:

$$\gamma = E[X] - \mu = \sqrt{\frac{2}{\pi}}(\sigma_2 - \sigma_1)\,, \tag{A6}$$

so that $\gamma > 0$ when the distribution is biased to the right and $\gamma < 0$ when the distribution is biased to the left.

The cumulative distribution function (cdf) for a TPN random variable is given by:

$$P[X \leq x] = \begin{cases} C \cdot \sigma_1\sqrt{2\pi} \cdot \Phi\left(\frac{x-\mu}{\sigma_1}\right) & for \quad x \leq \mu \\ 1 - C \cdot \sigma_2\sqrt{2\pi}\left[1 - \Phi\left(\frac{x-\mu}{\sigma_2}\right)\right] & for \quad x > \mu \end{cases}, \tag{A7}$$

and the α-th percentile[19] can be obtained from:

$$l = \begin{cases} \mu + \sigma_1\Phi^{-1}\left(\frac{\alpha}{C \cdot \sigma_1\sqrt{2\pi}}\right) & for \quad \alpha \leq p = P[x \leq \mu] \\ \mu + \sigma_2\Phi^{-1}\left(\frac{\alpha + C \cdot \sigma_2 \cdot \sqrt{2\pi} - 1}{C \cdot \sigma_2\sqrt{2\pi}}\right) & for \quad \alpha > p = P[x \leq \mu] \end{cases}. \tag{A8}$$

[19] For $0 < \alpha < 1$, the α-th percentile of the distribution is a constant l such that $P[X \leq l] = \alpha$.

Appendix B. Determining the triple

For the construction of the fan chart it is necessary to determine three parameters: $(\mu_h, \sigma_{1h}, \sigma_{2h})$ for each projection horizon $h = 1,2, \dots, H$. Having obtained the mode, skew and variance parameters for each horizon one needs to transform this set of information into $(\mu_h, \sigma_{1h}, \sigma_{2h})$. Combining (A6) with (A4) we obtain the system of two equations that can be solved for σ_{1h} and σ_{2h}.

$$\begin{cases} \gamma_h = \sqrt{\frac{2}{\pi}}(\sigma_{2h} - \sigma_{1h}) \\ \sigma_h^2 = \left(1 - \frac{2}{\pi}\right)(\sigma_{2h} - \sigma_{1h})^2 + \sigma_{1h}\sigma_{2h} \end{cases} \tag{B1}$$

Equations (B1) can be reduced to the following quadratic equation:

$$\sigma_{1h}^2 + b_h\sigma_{1h} + c_h = 0 \tag{B2}$$

where $b_h = \gamma_h/k$, $c_h = -[\left(1 - \frac{1}{k^2}\right)\gamma_h^2 + \sigma_h^2]$, and $k = \sqrt{\frac{2}{\pi}}$.

Equation (B2) puts some restrictions on combinations of variance and asymmetry of the TPN distribution, namely when it has complex solutions, then it is not possible to derive the TPN random variable with desired parameters. However, in majority of the case (B2) has two real-valued solutions, where one is typically negative, but in general one chooses the highest value root.

Appendix C. Highest probability density regions

Formally a region $A \in \Omega$ is a $100\alpha\%$ highest probability density region if it satisfies two following constraints:[20]

1) $P(x \in A) = \alpha$
2) $P(x) \geq P(y), \forall(x \in A, y \notin A)$

For the one-dimensional, unimodal probability density function $f_X(x)$, $100\alpha\%$ HPD region corresponds to the shortest interval $[x_l, x_u]$ such that $P[x_l \leq X \leq x_u] = \alpha$. It could be easily shown, that if the interval $[x_l, x_u]$ satisfies:

i. $\int_{x_l}^{x_u} f_X(x)dx = \alpha$
ii. $f_X(x_l) = f_X(x_u) > 0$
iii. $x_l \leq \mu \leq x_u$, where μ is the unique mode of $f_X(x)$

then it is the shortest interval satisfying (i).

For the TPN distribution specified by triple $(\mu, \sigma_1, \sigma_2)$, it is possible to determine the HPD region of size α as:

$$\begin{aligned} x_l &= \mu - \sigma_1 \frac{x_u - \mu}{\sigma_2} \\ x_u &= \mu + \sigma_2 \Phi^{-1}\left(\frac{1+\alpha}{2}\right) \end{aligned} \tag{C1}$$

where Φ^{-1} is the inverse of cdf of standard normal distribution.

[20] The first property is true for any $100\alpha\%$ region.

Appendix D. Variance estimations

The procedure of estimating projection variance is similar to that proposed by Blix and Sellin (1998) and Pinheiro and Esteves (2010). We assume that after a history of projection exercises we have data on projection errors e_h for each forecasted variable and each period of the projection $h = 1,2,\dots,H$.

The first step of the procedure is to compute the historical variance as function of horizon h. For the first period h=1 (and earlier periods in case of data revisions), for each endogenous variable the mean square errors are calculated. For future horizons, due to unequal amount of data variance is smoothed by estimating (using FGLS because of heteroscedascity) the following equation:

$$e_i^2 = \gamma_1 + \gamma_2 * \gamma_3^h + \varepsilon_i, \qquad \text{(D1)}$$

where $\gamma_1 > 0, \gamma_2, 0 < \gamma_3 < 1$. Fitted values from these equations are treated as an unbiased estimator of variance.

Since every projection exercise is run conditional to forecasted exogenous variables a set of deviations from the forecasted paths of the $m = 1,\dots,M$ variables is available:

$$\boldsymbol{z_h} = [z_{h,1} \;\; z_{h,2} \;\; \dots \;\; z_{h,M}]'. \qquad \text{(D2)}$$

An endogenous forecasting error u_h is defined as the portion of the deviation of endogenous variables from their forecasted paths which would exists even if the vector $\boldsymbol{z_h} = \boldsymbol{0}$.

The forecasters have a knowledge on the impact the unitary shocks to the exogenous variables have on the endogenous variables. Hence, for each endogenous variable they are able to construct a set of $(1xM)$ C_τ vectors of impulse responses for $\tau = 0,1,..,H-1$, where C_τ is a response at $t+\tau$ period for the shock occurring in period t for each exogenous variable.

The overall forecasting error, using linear approximation, can be decomposed as a weighted sum of the exogenous and endogenous errors:

$$e_h = \sum_{t=1}^{H} (C_{h-t} z_t + u_t) \quad \text{for } h = 1,2,\dots,H. \qquad \text{(D3)}$$

In matrix notation this can be written as:

$$\boldsymbol{e} = \boldsymbol{C} \cdot \boldsymbol{z} + \boldsymbol{u}, \tag{D4}$$

where $\boldsymbol{e} = [e_1, e_2, \dots, e_H]'$, $\boldsymbol{z}$ and $\mathbf{u}$ are defined in the same manner, and the matrix $\boldsymbol{C}$ is given by

$$\begin{bmatrix} C_0 & 0 & \dots & 0 \\ C_1 & C_0 & \dots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ C_{H-1} & C_{H-2} & \dots & C_0 \end{bmatrix} \tag{D5}$$

Then, assuming that the deviations of the exogenous variables from their predicted paths and the endogenous errors are uncorrelated, the variance of the forecasting errors is given by:

$$var(\boldsymbol{e}) = var(\boldsymbol{C} \cdot \boldsymbol{z} + \boldsymbol{u}) = var(\boldsymbol{C} \cdot \boldsymbol{z}) + var(\boldsymbol{u}) = \boldsymbol{C}var(\boldsymbol{z})\boldsymbol{C}' + var(\boldsymbol{u}) =$$
$$= \boldsymbol{CZC}' + var(\boldsymbol{u}). \tag{D6}$$

Since the economic conditions are changing from round to round, there is a possibility of introducing an expert judgement regarding the exogenous component of the variance. A diagonal matrix of scaling factors is defined as:

$$\boldsymbol{S} = diag_{h,j}(s_{h,j}) \tag{D7}$$

where $s_{h,j}$ is the scaling factor to the historical standard deviation of $z_{h,j}$, hence covariance matrix of the adjusted exogenous error becomes: $\boldsymbol{S} \cdot \boldsymbol{Z} \cdot \boldsymbol{S}$.

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