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Abstract

Analysis of macroeconomic expectations of private sector agents reveals not only the path of expected macroeconomic developments and the mechanism of expectation formation, but also the degree of uncertainty faced by economic agents. Relying on this observation the aim of this paper is twofold. First, we lay down the formal theory of subjective expectations and derive the subjective assessment hypothesis, which defines the optimal forecast of an individual under a specific loss function. Keeping the mathematical rigor, the proposed theory attempts to reflect the process of expectation formation that is consistent with well known behavioral features. Second, we apply the subjective assessment hypothesis to pin down the notion of uncertainty. We propose a novel uncertainty index, retrieved from forecast revisions of professional forecasters using an empirical model within the Bayesian approach. The two versions of this index derived for the US—based either on GDP growth or on inflation forecasts—describe different kinds of macroeconomic uncertainty. Their shocks act similarly as demand shocks, pushing economic activity and inflation down.

JEL: C10, C11, E31, D81, D84

Keywords: Subjective expectations, Uncertainty, Stochastic volatility, Identification, Bayesian approach, Survey of Professional Forecasters.

1 Introduction

The interest of economists in the role of uncertainty and what follows, in measurement of this phenomenon, has increased substantially since the global financial crisis in 2008/2009 and recently, as a result of the Covid-19 pandemic. However, any empirical research on the importance of uncertainty requires a reliable measure of this variable, which by its nature is latent. The economic literature offers a number of ways to proxy uncertainty, but a consensus on how to measure uncertainty is still far from being achieved. The common approaches rely on the volatility of stock market returns (Bloom, 2009), conditional volatility of forecast errors (Jurado et al., 2015) or frequency of appearances of newspaper's articles related to uncertainty (Baker et al., 2016). Other popular proxies of macroeconomic uncertainty are derived from survey-based subjective expectations and include disagreement among forecasters about future outcomes and the variance of subjective probability distributions of their forecasts (Zarnowitz and Lambros, 1987; Bomberger, 1996; Boero et al., 2008; Rich and Tracy, 2010).

The notion of uncertainty is indeed intimately related to people's expectations about the future. Therefore, in the present paper we set out a theory of subjective expectations and apply it to assess macroeconomic uncertainty. Our novel approach has strict theoretical foundations. The theory of subjective expectations is based on few realistic axioms and allows for omitting explicit reference to induced probability. In addition we augment the axioms with the subjective assessment hypothesis (SAH) which describes the optimal decision rule for each individual facing a forecasting task. The SAH assumes that the process of forming expectations involves conscious and self-reflected thinking, as well as a persistence of beliefs.

Our framework allows us to infer about uncertainty from the revisions of macroeconomic expectations. We decompose forecast revisions into two unobserved components: the "rational" part, driven by fundamentals, and the part associated with uncertainty. The former one is related to the information available for individuals when forming expectations, while the latter one reflects unexplainable part in subjective assessment of future outcomes (hunch, intuition, etc.). In order to make this decomposition operational and to retrieve the uncertainty from subjective expectations, we propose the empirical model with time-varying coefficients and stochastic volatility and provide conditions for its identifiability. Finally, we define the uncertainty index as the volatility of the part of revisions unexplained by fundamental factors.

In the empirical part of the paper we calculate two versions of the uncertainty index for the US economy based either on GDP growth or inflation forecast revisions in the Federal Reserve Bank of Philadelphia's Survey of Professional Forecasters (SPF). Empirical analysis suggests that both indices describe different kinds of macroeconomic uncertainty, while shocks to both indices act similarly as demand shocks, generating economic contractions and pushing inflation down.

The paper is organized as follows. First two sections introduce in a formal way the theory of subjective expectations and formulate the subjective assessment hypothesis. Section 4 links theoretical exposition and empirical application of the newly introduced theory, discussing how to elicit uncertainty from revisions in macroeconomic forecasts of the private sector. Section 5 is empirical. It presents the uncertainty indices for the US calculated over the period from mid-1981 to 2020 and shows impulse responses of selected macroeconomic variables to uncertainty shocks based on VAR models. The final section concludes.

2 Theory of subjective expectations

Quite realistically, we assume that all unknown quantities (in short $u.q.'s$) about which every individual or agent can form expectations are finite and real-valued. Let us denote the space of all such $u.q.'s$ as $\mathcal{B}$. From the mathematical point of view we may think of each element in $\mathcal{B}$ as an underlying function from some abstract space Ω into $\mathbb{R}$. In analogy to the subjective expected utility theory developed in Savage (1954), each $u.q.$ is treated as a function $\mathbf{x} := \{x(\omega)|\omega \in \Omega\}$, where ω is the state of the "world", understood as the context borne in mind by an individual when forming expectations at some particular moment of time.[1] The value of $\mathbf{x}$ at some $\omega \in \Omega$ i.e. $x(\omega)$, is referred to as the outcome. Hence, the given $u.q.$ is the set of all its outcomes and when some individual is confronted with the task of making expectation she considers different outcomes that may be realized for different states of the world. Note that the space Ω is the same for each $\mathbf{x}, \mathbf{y} \in \mathcal{B}$ (e.g. both consumption and GDP are functions on Ω). We do not take a stand whether Ω contains infinitely many elements or not, hence in general $\mathbf{x}$ may be infinite dimensional. Evidently $\mathcal{B} = \{\mathbf{x}| \sup_{\omega\in\Omega} |x(\omega)| < \infty\}$ is the linear (or vector) space where addition is defined as $\mathbf{x}+\mathbf{y} := x(\omega)+y(\omega)$, for each $\omega \in \Omega$ and for any $\alpha \in \mathbb{R}$; $\alpha\mathbf{x} := \alpha x(\omega)$, for each $\omega \in \Omega$, hence we sometimes call $u.q.'s$ just vectors. Define the linear subspace $\mathcal{S} \subset \mathcal{B}$. You may think of $\mathcal{S}$ as the span of an arbitrary subset of $u.q.'s$ which is feasible to be evaluated when making expectations (for some individual or a hypothetical agent at some period of time). The expression $\mathbf{x} \geq 0$ ($\mathbf{x} > 0$) means $x(\omega) \geq 0$ ($x(\omega) > 0$) for all $\omega \in \Omega$. In addition, $\inf \mathbf{x} := \inf_{\omega\in\Omega} x(\omega)$ and $\sup \mathbf{x} := \sup_{\omega\in\Omega} x(\omega)$. Two particular $u.q.'s$ will be of special importance: $\mathbf{1}$ denotes $x(\omega) = 1$ for all $\omega \in \Omega$, and $\mathbf{0}$ denotes $x(\omega) = 0$ for all $\omega \in \Omega$.

In what follows the term subjective expectation is reserved for mappings $E : \mathcal{S} \to \mathbb{R}$ that obey the following three axioms:

Axiom 1. $E(\mathbf{x}+\mathbf{y}) = E(\mathbf{x}) + E(\mathbf{y})$*; for all* $\mathbf{x}, \mathbf{y} \in \mathcal{S}$

Axiom 2. $\inf \mathbf{x} \leq E(\mathbf{x}) \leq \sup \mathbf{x}$*; for all* $\mathbf{x} \in \mathcal{S}$

Axiom 3. $\mathbf{1}$ *belongs to* $\mathcal{S}$

Axioms 1 and 2 explicitly appear in works by de Finetti (see e.g. de Finetti, 1974, p. 74), but in the context of the so-called previsions. As a clarifying remark, note that Axioms 1 and 2 should hold not only for some particular

[1] We keep the convention that all $u.q.'s$ are signified in bold letters.

$u.q.'s$, as seemingly indicated, but for any linear combination of $u.q.'s$ e.g. $\phi\mathbf{1} + \alpha\mathbf{x} + \beta\mathbf{y} + \cdots + \gamma\mathbf{z}$, for all $\phi, \alpha, \beta, \ldots, \gamma \in \mathbb{R}$ and all $\mathbf{x}, \mathbf{y}, \ldots, \mathbf{z} \in \mathcal{S}$, since such a combination is in $\mathcal{S}$. The Axiom 1 is hard to violate if an individual is self-reflective. The obvious caveat is that there is the implicit assumption that measurement units of all $u.q.'s$ under consideration allow for addition. At least there should be some way to express those $u.q.'s$ in the same measurement units e.g. growth rates in percentage points. As for the Axiom 2, it is very realistic, stating that the subjective expectation must be between the lowest possible and the greatest possible outcome. It implies $E(\mathbf{x}) \geq 0$; for $\mathbf{x} \geq 0$, which is important for uniqueness property of the SAH. The Axiom 3 is seemingly innocuous but in fact very important for our further reasoning. Taken together, the Axioms 2 and 3 imply $E(\mathbf{1}) = 1$. In addition, our axioms imply that (i) on every linear subspace $\mathcal{S} \subset \mathcal{B}$ there exists at least one subjective expectation, (ii) $E(\alpha\mathbf{x}) = \alpha E(\mathbf{x})$, for every $\alpha \in \mathbb{R}$, see e.g. Taylor (1958) p. 393, in particular $E(\mathbf{0}) = 0$ and (iii) $E(\alpha\mathbf{x} + \beta\mathbf{y} + \cdots + \gamma\mathbf{z}) = \alpha E(\mathbf{x}) + \beta E(\mathbf{y}) + \cdots + \gamma E(\mathbf{z})$, for all $\alpha, \beta, \ldots, \gamma \in \mathbb{R}$ and all $\mathbf{x}, \mathbf{y}, \ldots, \mathbf{z} \in \mathcal{S}$. In mathematical terms, each subjective expectation is the positive linear functional i.e. a linear functional with the additional property that $E(\mathbf{x}) \geq 0$, whenever $\mathbf{x} \geq 0$.

We claim that our axioms are descriptive. What distinguishes our axioms from those that may be found in the (huge) literature on choice under risk and uncertainty, is the treatment of the main objects. In our theory these objects are $u.q.'s$ whereas in the latter literature these are usually acts or objective probabilities. We intend to address the typical situation. A typical individual, even a professional forecaster, can form expectations concerning very few $u.q.'s$, which implies that $\mathcal{S}$ is very sparse. Hence we reject from the outset the postulate that the linear space under consideration may be freely expanded to accommodate "rankings" among any $u.q.'s$ including those confined to arbitrary small "events" in Ω. In contrast, in the literature on choice, an individual can usually accommodate rankings of a huge number of acts involving the rich class of subsets of Ω or form any mixtures of probabilities on outcomes. To put it another way, if we assume that an individual can form expectation of $\mathbf{x}$, it means that for this individual at the particular moment, this $\mathbf{x}$ is the given function (but any bounded mapping from Ω into $\mathbb{R}$), to which she applies her subjective expectation. We think that our position in this respect conforms to the actualist rationality put forward by Manski (2011), who describes it succinctly stating that "*Unlike Savage, our agent does not seek consistency of behavior across hypothetical choice* *sets. He just wants to make a reasonable choice from the one choice set that he actually faces*".

Let us call $\mathcal{S}_0 = \{\alpha\mathbf{1}|\alpha \in \mathbb{R}\} \subset \mathcal{S}$ the subspace of all known quantities (in short $k.q.'s$). These quantities are known since $E(\alpha\mathbf{1}) = \alpha$, for all subjective expectations, hence whatever the state of the world will be realized, the outcome is the same and equals α. The following proposition is simple yet important.

Proposition 1. *(Uniqueness) For each individual there is one and only one subjective expectation on any* $\mathcal{S} \subset \mathcal{B}$.

The axioms and the above proposition allow us to focus on subjective expectations that abstract from probability notion, which we find convenient and realistic. In our approach, when an individual faces the problem of the expectation formation, she sizes up the whole situation, considers future scenarios that come to her mind (relevant ω's and their outcomes) and ultimately chooses one outcome from $\mathbf{x}$ that represents her subjective expectation.

There are two reasons which motivate the omission of the notion of probability in our framework. First, as pointed out by some authors, expressing expectations by economic agents in the form of a probabilistic distribution is a very complex task and might lead to unreliable outcomes. For example Hogarth (1975) casts doubts on the possibility to express subjective probability distributions by experts: *"In summary, man is a selective, stepwise information processing system with limited capacity, and, as I shall argue, he is ill-equipped for assessing subjective probability distributions. Furthermore, man frequently just ignores uncertainty."* Winkler (1967) is even more skeptical and states: *"It must be stressed that the assessor has no built-in prior distribution which is there for the taking. That is, there is no "true" prior distribution. Rather, the assessor has certain prior knowledge which is not easy to express quantitatively without careful thought. An elicitation technique used by the statistician does not elicit a "true" prior distribution, but in a sense helps to draw out an assessment of a prior distribution from the prior knowledge. Different techniques may produce different distributions because the method of questioning may have some effect on the way the problem is viewed."* A comprehensive review of challenges in eliciting subjective probabilities, including a survey of heuristics and biases commonly found in the responses to questions about uncertain quantities, is presented in Garthwaite et al. (2005).

Second, we prefer modeling subjective expectations without referring to the induced probability, because in this way we avoid making unrealistic assumptions needed to establish the 1-1 correspondence between these two objects. Using some further assumptions one might demonstrate that there is an isometric isomorphism (i.e. a mapping that is linear, 1-1, onto and norm preserving) between subjective expectation and finitely additive probability measure[2], and hence it is perfectly justified to focus on modeling either subjective expectations or the induced probability (from the mathematical point of view they are indistinguishable). Unfortunately these “further assumptions” are not realistic enough to be accepted uncritically (in contrast to our axioms). Potential objections may be grouped into three categories: (i) measurability requirements for unknown quantities; (ii) imposing “richness” of the linear space $\mathcal{S}$ in relation to the states of the world (i.e. Ω); (iii) the finite additiveness of the induced probability. Although objection (iii) may be overcome with some additional assumptions (i.e. at the cost of further losing the realisticness of the setup one may get the countable additivity), the first two ones are rather fundamental. These details are fully worked out by the authors and available on the request.

[2]Using the theorem which is essentially the version presented in Rao and Rao (1983), p. 135.

3 The subjective assessment hypothesis

Our methodological position is that all we have is the subjective expectation and our goal is to suggest some mathematical theory to deal with it more formally. In order to do this effectively, we need to allow for more operations on elements in $\mathcal{B}$. To this end we assume that $\mathcal{B}$ is not only the linear space, but also the algebra, where the multiplication is defined as $\mathbf{x} \cdot \mathbf{y} := x(\omega)y(\omega)$, for each $\omega \in \Omega$. This is both mandatory (since $\sup_{\omega\in\Omega} |x(\omega)y(\omega)| \leq \sup_{\omega\in\Omega} |x(\omega)| \sup_{\omega\in\Omega} |y(\omega)| < \infty$) and realistic assumption (think of making expectations for real variables times the underlying deflator i.e. nominal variables). In any case, we need to assume that if an individual can form expectations for both $\mathbf{x}$ and $\mathbf{y}$, then she can (upon some reflection) provide expectations of $\mathbf{x}\cdot\mathbf{y}$. Let us define the subalgebra $\mathcal{S} \subset \mathcal{B}$. The fact that $\mathcal{S}$ is also the linear subspace justifies the abuse of notation introduced in section 2. Now you may think of $\mathcal{S}$ as the span of an arbitrary subset of $u.q.'s$ together with a product of every pair of these $u.q.'s$, which is feasible to be evaluated when making expectations (for some individual at some period of time). Extending permissible operations on $u.q.'s$ allows us to get e.g. $E(\alpha\mathbf{x}\cdot(\mathbf{y}+\beta\mathbf{z})) = E(\alpha\mathbf{x}\cdot\mathbf{y}+\alpha\beta\mathbf{x}\cdot\mathbf{z}) = \alpha E(\mathbf{x}\cdot\mathbf{y})+\alpha\beta E(\mathbf{x}\cdot\mathbf{z})$, for all $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathcal{S}$ and all $\alpha, \beta \in \mathbb{R}$.

In order to push the theory a little bit further we need to define the distance between elements in $\mathcal{S}$ i.e. to find a norm on $\mathcal{S}$. To this end let us denote $\|\mathbf{x}\| := \sqrt{E(\mathbf{x}\cdot\mathbf{x})}$ and from now on, we refer to $\|\mathbf{x}\|$ as the subjective norm.[3] We define the subjective expected distance between two $\mathbf{x},\mathbf{y} \in \mathcal{S}$ as $\|\mathbf{x}-\mathbf{y}\|^2 := E((\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}))$. In particular we allow for possibility that either $\mathbf{x}$ or $\mathbf{y}$ may be a $k.q.$

Let us define the null space of the subjective expectation as $\mathcal{N}(E) = \{\mathbf{x} \in \mathcal{S} | E(\mathbf{x}) = 0\}$. Note that $\mathcal{N}(E)$ is a proper subset of $\mathcal{S}$ since $E(\mathbf{1}) = 1$. Let $\hat{\mathbf{x}} \in \mathcal{S}$ be any fixed $u.q.$ that does not belong to $\mathcal{N}(E)$. Then each $\mathbf{x} \in \mathcal{S}$ may be written as $\mathbf{x} = \phi\hat{\mathbf{x}} + \eta$, where $\eta \in \mathcal{N}(E)$, so that $E(\mathbf{x}) = \phi E(\hat{\mathbf{x}})$. Moreover such a representation is unique i.e. if $\mathbf{x} = \phi\hat{\mathbf{x}} + \eta = \bar{\phi}\hat{\mathbf{x}} + \bar{\eta}$, then $\phi = \bar{\phi}$ and $\eta = \bar{\eta}$.

[3] The norm on $\mathcal{S}$ is a real-valued function mapping each $\mathbf{x} \in \mathcal{S}$ into $\|\mathbf{x}\|$ with the following three properties 1) $\|\mathbf{x}\| \geq 0$, $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$, 2) $\|\alpha\mathbf{x}\| = |\alpha| \|\mathbf{x}\|$ for any $\alpha \in \mathbb{R}$ and 3) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$, for all $\mathbf{x},\mathbf{y} \in \mathcal{S}$. It is worth stressing that in order to prove that the subjective norm is indeed a norm we need only to assume our three axioms and the multiplication operation on $u.q.'s$.

To be a little more constructive one may write:

$$\mathbf{x} = \underbrace{\frac{E(\mathbf{x})}{E(\hat{\mathbf{x}})}}_{\phi}\hat{\mathbf{x}} + \underbrace{\mathbf{x} - \frac{E(\mathbf{x})}{E(\hat{\mathbf{x}})}\hat{\mathbf{x}}}_{\eta} \tag{1}$$

So far we abstracted from the time at which expectations are made and how they are constructed in general. The timing is necessary to come up with some dynamic formulas. Let us denote $X_t = \{\mathbf{1}, \mathbf{x}_{1,t}, \mathbf{x}_{2,t}, \ldots, \mathbf{x}_{n,t}\}$, where $\mathbf{x}_{i,t}$ is the $i-$th $u.q.$ (out of n) that may be attributed to the time t. Denote $\mathcal{S}_t \subseteq \mathcal{S}$ as the subalgebra generated by X_t. Let us put $\hat{\mathbf{x}} = \mathbf{1} + \alpha_1\mathbf{x}_{1,t} + \cdots + \alpha_n\mathbf{x}_{n,t} \in \mathcal{S}_t$, for any $\alpha_i \in \mathbb{R}$. Then (1) may be written as

$$\mathbf{x} = \phi\mathbf{1} + \beta_1\mathbf{x}_{1,t} + \cdots + \beta_n\mathbf{x}_{n,t} + \eta \tag{2}$$

where $\beta_i = \phi\alpha_i$. Since equation (2) is valid for any $\mathbf{x} \in \mathcal{S}$, we can choose any $u.q.$ from X_{t+1} (except $\mathbf{1}$, which would result in triviality). Let us denote it as $\mathbf{y}_{t+1}$. Then we arrive at the formula strikingly similar to the linear regression (but involving $u.q.'s$)

$$\mathbf{y}_{t+1} = \phi\mathbf{1} + \beta_1\mathbf{x}_{1,t} + \cdots + \beta_n\mathbf{x}_{n,t} + \eta \tag{3}$$

Let us denote $\hat{\mathbf{y}}_{t+1} = \phi\mathbf{1} + \beta_1\mathbf{x}_{1,t} + \cdots + \beta_n\mathbf{x}_{n,t}$.[4] We should emphasize that (3) is the identity that follows directly from (1). Even if (3) has nothing to do with the causality, it might be useful. If we impose the (subjective) rule that picks coefficients $\phi, \beta_1, \ldots, \beta_n$, then this amounts to the same thing as modeling $\mathbf{y}_{t+1}$ in relation to $\mathbf{1}, \mathbf{x}_{1,t}, \mathbf{x}_{2,t}, \ldots, \mathbf{x}_{n,t}$. Implicitly, those chosen $\mathbf{x}'_{i,t}s$ have to be relevant for $\mathbf{y}_{t+1}$ and should constitute its contextual "economic world".

In addition let us attach the subscript h to E, which means that subjective expectation is made at this time. This concerns both expression (1) and the subjective norm, hence we introduce the notation $\|\mathbf{x}\|_h^2 := E_h(\mathbf{x} \cdot \mathbf{x})$ (i.e the squared subjective norm evaluated at time h), frequently used hereafter. Then we are in a position to close our theoretical framework by complementing our (descriptive) three axioms with the descriptive assumption.

Assumption 1. *(Decision rule) Each individual makes her subjective assessment at time* $h \leq t+1$ *consistent with the minimization* $\|\mathbf{y}_{t+1} - \hat{\mathbf{y}}_{t+1}\|_h^2 + \lambda\,\|\bar{y}_h\mathbf{1} - \hat{\mathbf{y}}_{t+1}\|_h^2$ *with respect to* $\phi, \beta_1, \ldots, \beta_n$*, where* $\lambda \geq 0$ *and* $\bar{y}_h \in \mathbb{R}$ *are fixed.*

[4] From now on, hatted $u.q.'s$ indicate prime objects to be chosen by an individual. They are fully characterized by the coefficients $\phi, \beta_1, \ldots, \beta_n$.

In line with this assumption, an individual chooses $u.q.$ $\hat{\mathbf{y}}_{t+1}$, such that the subjective expected (joint) distance from $\mathbf{y}_{t+1}$ and $\bar{y}_h \mathbf{1} \in \mathcal{S}_0$ is as small as possible. The subscript h in $\bar{y}_h$ signifies that it may be the time-varying constant, but nevertheless it is known at the moment the subjective expected distance is evaluated i.e. at time h. In particular $\bar{y}_h$ might be equal to past (realized) values of quantities about which one forms expectations, individual's past beliefs about these quantities or some publicly known forecasts.

We call the minimizing function in Assumption 1 as the decision rule. The first term in this rule will be referred to as the quadratic term and the second one – the penalty. The quadratic term is the abstract counterpart of quadratic functions which are very popular in economic literature. In fact, assumption that forecasters minimize quadratic forecast error has long tradition in the rational expectations literature (e.g., Pesando, 1975; Modigliani and Shiller, 1973; Keane and Runkle, 1990; Zarnowitz, 1985). However, there are strong theoretical and empirical premises to consider additional terms in the decision rule. This is embraced by our penalty term. This is probably the simplest possible way to capture many theories and approaches that appeared in the broad literature. For example, interpreting $\bar{y}_h$ as lagged consensus or forecasters' previous prediction is in line with models of strategic behavior (Ehrbeck and Waldmann, 1996; Ottaviani and Sorensen, 2006; Trueman, 1994) and literature on cognitive limitations of forecasters (Tversky and Kahneman, 1974; Coibion and Gorodnichenko, 2015, and literature mention therein).[5] Alternatively, in the case of inflation expectations, referring to learning from experience models leads to interpreting $\bar{y}_h$ as past history of inflation (e.g. Goldfayn-Frank and Wohlfart, 2020; Malmendier and Nagel, 2016), while other strand of literature points that central bank communication, like inflation target or inflation projection, provides anchors for inflation expectations (e.g., Beechey et al., 2011).

Assumption 1 allows us to come up with the optimal decision rule for an individual (i.e. a solution to the decision rule). We describe this rule in proposition 2 and call it the subjective assessment hypothesis (SAH).

Proposition 2. *(SAH) Under Assumption 1,*

$\hat{\mathbf{y}}_{t+1} = (1-\alpha)[E_h(\mathbf{y}_{t+1})\mathbf{1}+\beta_{1,h}(\mathbf{x}_{1,t}-E_h(\mathbf{x}_{1,t})\mathbf{1})+\cdots+\beta_{n,h}(\mathbf{x}_{n,t}-E_h(\mathbf{x}_{n,t})\mathbf{1})]+\alpha\bar{y}_h\mathbf{1}$,

[5] Forecast smoothing and reliance on past forecasts is confirmed in numerous empirical studies on wide range of forecasted variables (e.g., Clement and Tse, 2005; Deschamps and Ioannidis, 2013; Fujiwara et al., 2013; Fuhrer, 2018; Nakazono, 2013; Nordhaus and Durlauf, 1984; Nordhaus, 1987; Peterson, 2001).

where $0 \leq \alpha = \frac{\lambda}{1+\lambda} < 1$ *and* $\beta_{1,h}, \ldots, \beta_{n,h}$ *are given by the solution to the following system of linear equations (in* $\beta_1, \ldots, \beta_n$*):*

$$\underbrace{\begin{bmatrix} var_h(\mathbf{x}_{1,t}) & cov_h(\mathbf{x}_{1,t}, \mathbf{x}_{2,t}) & \ldots & cov_h(\mathbf{x}_{1,t}, \mathbf{x}_{n,t}) \\ cov_h(\mathbf{x}_{1,t}, \mathbf{x}_{2,t}) & var_h(\mathbf{x}_{2,t}) & \ldots & cov_h(\mathbf{x}_{2,t}, \mathbf{x}_{n,t}) \\ \vdots & \vdots & \ddots & \vdots \\ cov_h(\mathbf{x}_{1,t}, \mathbf{x}_{n,t}) & cov_h(\mathbf{x}_{2,t}, \mathbf{x}_{n,t}) & \ldots & var_h(\mathbf{x}_{n,t}) \end{bmatrix}}_{W_h} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} = \begin{bmatrix} cov_h(\mathbf{y}_{t+1}, \mathbf{x}_{1,t}) \\ cov_h(\mathbf{y}_{t+1}, \mathbf{x}_{2,t}) \\ \vdots \\ cov_h(\mathbf{y}_{t+1}, \mathbf{x}_{n,t}) \end{bmatrix},$$

where $var_h(\mathbf{x}_{i,t}) := E_h[(\mathbf{x}_{i,t} - E_h(\mathbf{x}_{i,t})\mathbf{1}) \cdot (\mathbf{x}_{i,t} - E_h(\mathbf{x}_{i,t})\mathbf{1})]$,
$cov_h(\mathbf{x}_{i,t}, \mathbf{x}_{j,t}) := E_h[(\mathbf{x}_{i,t} - E_h(\mathbf{x}_{i,t})\mathbf{1}) \cdot (\mathbf{x}_{j,t} - E_h(\mathbf{x}_{j,t})\mathbf{1})]$, *and* $cov_h(\mathbf{y}_{t+1}, \mathbf{x}_{i,t})$ *is defined analogously.*

The SAH has several important features. Firstly, note that there appears the time subscript h among coefficients $\beta_{1,h}, \ldots, \beta_{n,h}$. It follows from the above system of equations that these parameters must evolve in time. Secondly, the solution with respect to $\beta_1, \ldots, \beta_n$ is unique iff W_h is nonsingular. In general, this happens if from the subjective perspective at time h, $\mathbf{x}'_{i,t}s$ are $u.q.'s$. Otherwise the entries in W_h may be zeros and (all or some of) $\beta_{1,h}, \ldots, \beta_{n,h}$ may be undetermined. Thirdly, the notions for variances and covariances are self-explained generalizations of the standard statistics defined on scalar values. However keep in mind that in our case these notions are defined on functions from abstract space Ω into $\mathbb{R}$ i.e. $u.q.'s$, without any reference to the probability. Lastly, irrespective of whether W_h is nonsingular or not, we have $E_h(\hat{\mathbf{y}}_{t+1}) = (1-\alpha)E_h(\mathbf{y}_{t+1}) + \alpha\bar{y}_h$. It means two things: (i) since $\hat{\mathbf{y}}_{t+1}$ is the best subjective assessment, what is observed is its subjective expectation $E_h(\hat{\mathbf{y}}_{t+1})$ (not $E_h(\mathbf{y}_{t+1})$) and (ii) $E_h(\mathbf{y}_{t+1})$ is unobservable unless $\alpha = 0$, in which case $E_h(\hat{\mathbf{y}}_{t+1}) = E_h(\mathbf{y}_{t+1})$. Since $\alpha = 0$ iff $\lambda = 0$, the role of the penalty term in the decision rule should now be fully appreciated.

In the special case, when $\lambda = 0$ in the decision rule, uniqueness of $\hat{\mathbf{y}}_{t+1}$ follows from standard projection theorems, otherwise it may not be the case. The nice feature of our setup is the fact that even if $\beta_{1,h}, \ldots, \beta_{n,h}$ may be nonunique, we have the following proposition:

Proposition 3. *Under Assumption 1,* $\hat{\mathbf{y}}_{t+1}$ *is unique.*

Hence bearing in mind Propositions 3 and 1, there is no ambiguity concerning both subjective assessment $\hat{\mathbf{y}}_{t+1}$ (which is unobserved) and its "trace" $E_h(\hat{\mathbf{y}}_{t+1})$ (which is observed).

4 Revision of expectations and uncertainty

4.1 Decomposition of change in expectations

In this section we use the theory of subjective expectations to analyze revisions of expectations between two periods. We consider difference between nowcasting at time $t+1$ and one-period-ahead expectation made at time t, i.e. $E_{t+1}(\hat{\mathbf{y}}_{t+1}) - E_t(\hat{\mathbf{y}}_{t+1}) = (1-\alpha)(E_{t+1}(\mathbf{y}_{t+1}) - E_t(\mathbf{y}_{t+1})) + \alpha(\bar{y}_{t+1} - \bar{y}_t)$. We show that this revision of expectations may be decomposed into "rational" and uncertainty components.

Since our ultimate goal is to apply our methods to the real world problem (i.e. expectations from the SPF), in what follows we assume that at time t, all $\mathbf{x}'_{i,t}s$ are $u.q.'s$ (not $k.q.'s$). On the other hand, given that the SAH describes the optimal decision rule for each individual and is valid for all $\mathbf{x}'_{i,t}s$ in $\mathcal{S}$, we might consider a hypothetical case, in which an individual, just after obtaining the SAH, gets to know the true $\mathbf{x}'_{i,t}s$. By this we mean replacing $x_{i,t}\mathbf{1} \in \mathcal{S}_0$ for $\mathbf{x}_{i,t}$, but leaving $E_h(\mathbf{x}_{i,t})$ untouched. Then taking subjective expectation evaluated at time $h = t$ to both sides in the SAH we arrive at the hypothetical subjective expectation (we mark such expectations with tilde):

$$\widetilde{E}_t(\hat{\mathbf{y}}_{t+1}) = \tag{4}$$

$$= (1-\alpha)[E_t(\mathbf{y}_{t+1}) + \beta_{1,t}(x_{1,t} - E_t(\mathbf{x}_{1,t})) + \cdots + \beta_{n,t}(x_{n,t} - E_t(\mathbf{x}_{n,t}))] + \alpha\bar{y}_t$$

Needless to say $\widetilde{E}_t(\hat{\mathbf{y}}_{t+1})$ is unobserved. Then we can write

$$E_{t+1}(\hat{\mathbf{y}}_{t+1}) - E_t(\hat{\mathbf{y}}_{t+1}) = \tag{5}$$

$$= \underbrace{\widetilde{E}_t(\hat{\mathbf{y}}_{t+1}) - E_t(\hat{\mathbf{y}}_{t+1})}_{rational\,component} + \underbrace{E_{t+1}(\hat{\mathbf{y}}_{t+1}) - \widetilde{E}_t(\hat{\mathbf{y}}_{t+1})}_{uncertainty}$$

$$= (1-\alpha)\beta_{1,t}(x_{1,t} - E_t(\mathbf{x}_{1,t})) + \cdots + (1-\alpha)\beta_{n,t}(x_{n,t} - E_t(\mathbf{x}_{n,t})) + \varepsilon_{t+1}$$

where $\varepsilon_{t+1} = E_{t+1}(\hat{\mathbf{y}}_{t+1}) - \widetilde{E}_t(\hat{\mathbf{y}}_{t+1})$ will be referred to as the carrier of uncertainty, since it reflects some changes in expectations that are hard to justify even on the basis of the best opinion formed under the hypothetical extra knowledge of $\mathbf{x}'_{i,t}s$. To the extent that $E_{t+1}(\hat{\mathbf{y}}_{t+1}) = \widetilde{E}_{t+1}(\hat{\mathbf{y}}_{t+1})$,[6] the carrier of uncertainty reflects a hunch or intuition on future prospects of the economy, experienced between two

[6] This holds provided that $\mathbf{x}_{i,t}$, for $i = 1, \ldots, n$, is a $k.q.$ for each forecaster from the time perspective $t+1$, which should be reasonable assumption given the timing. Hence $E_{t+1}(\hat{\mathbf{y}}_{t+1})$ implicitly assumes that $\mathbf{x}'_{i,t}s$ are already revealed.

periods rather than the knowledge of the realized $\mathbf{x}'_{i,t}s$.

Equation (5) is not operational since $E_t(\mathbf{x}_{i,t})$ is not observed. In order to find the relevant expression note that by Assumption 1, $\|\mathbf{y}_t - \hat{\mathbf{y}}_t\|_t^2 + \lambda \|\bar{y}_t \mathbf{1} - \hat{\mathbf{y}}_t\|_t^2$. Putting $\mathbf{y}_t = \mathbf{x}_{i,t}$, $\hat{\mathbf{y}}_t = \hat{\mathbf{x}}_{i,t} = \phi\mathbf{1} + \beta_1\mathbf{x}_{1,t-1} + \cdots + \beta_n\mathbf{x}_{n,t-1}$, $\bar{y}_t = \bar{x}_{i,t}$, we get $E_t(\hat{\mathbf{x}}_{i,t}) = (1-\alpha)E_t(\mathbf{x}_{i,t}) + \alpha\bar{x}_{i,t}$. Plugging the latter into (5) we find

$$E_{t+1}(\hat{\mathbf{y}}_{t+1}) - E_t(\hat{\mathbf{y}}_{t+1}) = \beta_{1,t}(x_{1,t} - E_t(\hat{\mathbf{x}}_{1,t})) + \cdots + \beta_{n,t}(x_{n,t} - E_t(\hat{\mathbf{x}}_{n,t})) \quad (6)$$

$$-\alpha\beta_{1,t}(x_{1,t} - \bar{x}_{1,t}) - \cdots - \alpha\beta_{n,t}(x_{n,t} - \bar{x}_{n,t}) + \varepsilon_{t+1}$$

which operates on observables.

Note that (6) is the equation valid for an individual. Suppose we have k individuals. If we aggregate them all, it yields

$$\frac{1}{k}\sum E^i_{t+1}(\hat{\mathbf{y}}_{t+1}) - \frac{1}{k}\sum E^i_t(\hat{\mathbf{y}}_{t+1}) = \quad (7)$$

$$= \frac{1}{k}\sum \beta^i_{1,t}(x_{1,t} - E^i_t(\hat{\mathbf{x}}_{1,t})) + \cdots + \frac{1}{k}\sum \beta^i_{n,t}(x_{n,t} - E^i_t(\hat{\mathbf{x}}_{n,t})) +$$
$$-\frac{1}{k}\sum \alpha^i\beta^i_{1,t}(x_{1,t} - \bar{x}^i_{1,t}) - \cdots - \frac{1}{k}\sum \alpha^i\beta^i_{n,t}(x_{n,t} - \bar{x}^i_{n,t}) + \frac{1}{k}\sum \varepsilon^i_{t+1}$$

where the "i" superscript signifies terms specific for the $i-$th individual and $\sum := \sum_{i=1}^k$. Adding and subtracting appropriate terms we arrive at the "mean" equation for all individuals, which still takes the form of (6) except the appearance of a constant c_t, being a time-varying and quite complicated function of terms at the individual level. In general this is unimportant for economic reasoning but we have to take it into account in the estimation to avoid the problem of omitted variables.

In our framework it is natural to treat the volatility of the carrier of uncertainty as the uncertainty index. Since uncertainty may vary over time it is important to accommodate its time-varying nature. In addition, since by Proposition 2 all coefficients in (6) should be time-varying too, in order to model successfully the uncertainty index we employ regression with time-varying coefficients and stochastic volatility.

4.2 The empirical model of uncertainty index

In what follows we consider the model

$$y_t = c_t + (x_t - E_t(\hat{\mathbf{x}}))\beta_t - \alpha(x_t - \bar{x}_t)\beta_t + \varepsilon_t; \qquad t = 1, \dots, T \tag{8}$$

where

$$\begin{aligned}
y_t =& \frac{1}{k}\sum E^i_{t+1}(\hat{\mathbf{y}}_{t+1}) - \frac{1}{k}\sum E^i_t(\hat{\mathbf{y}}_{t+1}),\\
\beta_t =& (\frac{1}{k}\sum \beta^i_{1,t}, \dots, \frac{1}{k}\sum \beta^i_{n,t})',\\
x_t =& (x_{1,t}, \dots, x_{n,t}),\\
E_t(\hat{\mathbf{x}}) =& (\frac{1}{k}\sum E^i_t(\hat{\mathbf{x}}_{1,t}), \dots, \frac{1}{k}\sum E^i_t(\hat{\mathbf{x}}_{n,t})),\\
\alpha =& \frac{1}{k}\sum \alpha^i,\\
\bar{x}_t =& (\frac{1}{k}\sum \bar{x}^i_{1,t}, \dots, \frac{1}{k}\sum \bar{x}^i_{n,t}),\\
\varepsilon_t =& \frac{1}{k}\sum \varepsilon^i_{t+1},
\end{aligned}$$

which is completed by the following specification:

$$\varepsilon_t \sim N(0, e^{h_t}) \tag{9}$$

$$h_t = \mu + \psi(h_{t-1} - \mu) + u_t; \qquad u_t \sim N(0, \sigma^2) \tag{10}$$

$$\begin{bmatrix} c_t \\ \beta_t \end{bmatrix} = \nu + F\left(\begin{bmatrix} c_{t-1} \\ \beta_{t-1} \end{bmatrix} - \nu\right) + \omega_t; \qquad \omega_t \sim N(0, \Sigma) \tag{11}$$

where ω_t is independent of u_k for all t, k, and Σ is positive definite. Since the equation (8) suggests stationarity we assume $|\psi| < 1$ and F is diagonal i.e. $F = diag(f_1, f_2, \dots, f_{n+1})$, with $|f_i| < 1$ and $f_i \neq 0$, for all i.[7] In addition for the initial states we assume $h_0 \sim N(\mu, \underline{f}_1\sigma^2)$, $\begin{bmatrix} c_0 \\ \beta_0 \end{bmatrix} \sim N(\nu, \underline{f}_2\Sigma)$, where $\underline{f}_1, \underline{f}_2$ are two scalar hyperparameters (i.e. must be chosen before estimation).[8] Since we

[7]Diagonality of F is only for parsimony, which in time-varying model setup is highly desirable. However identification holds provided that F is stable and nonsingular matrix. The latter imposes the restriction $f_i \neq 0$, when F is diagonal.

[8]We depart from the mainstream in setting covariances of the initial states as $\underline{f}_1\sigma^2$ and $\underline{f}_2\Sigma$ instead of the implicit unconditional covariances $\frac{\sigma^2}{1-\psi^2}$ and $unvec\{(I - F \otimes F)^{-1}vec(\Sigma)\}$, respectively, where $unvec\{\cdot\}$ is the operation of "unvectorization". Our assumptions make all conditional posteriors in the Gibbs sampler standard.

follow the Bayesian approach we should also come up with the prior distribution for model parameters i.e. $\alpha, \sigma^2, \mu, \psi, \Sigma, \nu, F$ (see Appendix A.5 for details).

Since we are unaware of the relevant work that deals with the identification of the model setup (8)-(11), we provide conditions for its identifiability.[9] In what follows we assume that $T \to \infty$ (for the effect of the initial conditions to die away). Hence our results to be stated right below, refer to the so-called asymptotic identifiability.[10]

Proposition 4. *Under assumptions:*

1) F is nonsingular and all eigenvalues of F are strictly less than 1 in modulus,

2) Define $\tilde{z}_t = [1 \vdots x_t - E_t(\hat{\mathbf{x}}) \vdots \bar{x}_t - x_t]$. Let $\tilde{Z}_t = (\tilde{z}_1', \tilde{z}_2', \ldots, \tilde{z}_T')'$ be of full column rank and at least one of the unconditional means of β_t be nonzero,

3) Define $z_t = [1 \vdots x_t - E_t(\hat{\mathbf{x}}) - \alpha(x_t - \bar{x}_t)]$. Let $\begin{bmatrix} z_1 \otimes z_2 \\ z_2 \otimes z_3 \\ z_3 \otimes z_4 \\ \vdots \\ z_{T-1} \otimes z_T \end{bmatrix}$ *and* $\begin{bmatrix} z_1 \otimes z_3 \\ z_2 \otimes z_4 \\ z_3 \otimes z_5 \\ \vdots \\ z_{T-2} \otimes z_T \end{bmatrix}$ *have full column rank for all $0 \leq \alpha < 1$,*

All parameters in the model (8)-(11) are uniformly globally identified.

It's worth emphasizing that identification is extremely important for our construction of the uncertainty index since it guarantees that the data can distinguish between two sources of variability i.e. those attributed to time-varying coefficients and time-varying volatilities.

[9] We should mention the similarity of our identification conditions to those in Pagan (1980), which however abstract from stochastic volatility.

[10] This is only to simplify identification treatment. We could provide conditions for given sample size T but they do not change considerably the assumptions needed for identification. As a matter of fact, since our identification analysis is based on 1-1 correspondence between population data moments and model parameters, the case of large number of observations guarantees existence of reliable estimators for data moments.

5 Uncertainty index

5.1 Interpretation of uncertainty indices

We apply the model described in the previous section to measure uncertainty in the US. The uncertainty is captured by the uncertainty index defined as the time-varying standard deviation of the uncertainty carrier i.e. $\{e^{\frac{1}{2}h_t}\}$.

The proposed uncertainty index has several advantages over alternative available measures of uncertainty derived from surveys. Disagreement of forecasters is easily observable, but as pointed out by Zarnowitz and Lambros (1987) the concept of disagreement differs from uncertainty (although they are positively correlated). Contrary to forecast disagreement, our index has strong theoretical foundations as it is derived from the SAH. In comparison to measures of uncertainty calculated as variance of subjective predictive probability distributions, our approach does not require the assumption that people are capable of expressing expectations in the form of reliable distribution function. We believe that such assumption is too strong and unrealistic – see section 2. In our theory of subjective expectations probability appears only implicitly.

Employing survey data from the SPF we calculate two uncertainty indices based on forecasts of main macroeconomic aggregates: GDP growth and inflation (GDP deflator). We assume that forecasters consider also interest rate as an important input in their forecast, and we interpret $(\bar{y})$ in the decision rule (Assumption 1) as survey consensus from the previous period. In line with model (8), revisions of one-quarter ahead forecasts, referring to GDP growth or inflation, depend on surprise part of recent realizations of macroeconomic variables and their deviations from the anchor, i.e. consensus forecast. More specifically, empirical specification of equation (8) for GDP growth forecast revisions is the following:

$$\begin{aligned} g_{t+1|t+1} - g_{t|t+1} = c_t + \beta_{1,t}(\pi_t - \pi_{t|t}) + \beta_{2,t}(g_t - g_{t|t}) + \beta_{3,t}(i_t - i_{t|t}) + \\ - \alpha\left(\beta_{1,t}\left(\pi_t - \pi_{t-1|t}\right) + \beta_{2,t}\left(g_t - g_{t-1|t}\right) + \beta_{3,t}\left(i_t - i_{t-1|t}\right)\right) + \varepsilon_t \end{aligned} \tag{12}$$

where $g_{t_1|t_2}$ denotes forecast of GDP growth rate for quarter t_2 formulated by professional forecasters in quarter t_1, $\pi_{t_1|t_2}$ refers to SPF forecast of inflation, $i_{t_1|t_2}$ refers to SPF forecast of 3-month Treasury bill rate, and g_t, π_t, i_t denote, respectively, actual values of GDP growth, inflation rate and interest rate in period t. The analogous formula for inflation forecast revisions is given by:

$$\pi_{t+1|t+1} - \pi_{t|t+1} = c_t + \beta_{1,t}(\pi_t - \pi_{t|t}) + \beta_{2,t}(g_t - g_{t|t}) + \beta_{3,t}(i_t - i_{t|t}) + \\ -\alpha\left(\beta_{1,t}\left(\pi_t - \pi_{t-1|t}\right) + \beta_{2,t}\left(g_t - g_{t-1|t}\right) + \beta_{3,t}\left(i_t - i_{t-1|t}\right)\right) + \varepsilon_t \quad (13)$$

In both cases the evolution of the error term is described by (9)-(11).

Survey data and macro variables come from Real-Time Data Research Center of the Federal Reserve Bank of Philadelphia. All survey data are taken at aggregate level and represent means across forecasters. GDP growth and inflation refer to the annualized quarter-over-quarter percent change. We pay close attention to the timing of the survey and data releases to make sure that forecasters formulate their nowcasts before the full data for a given quarter is known. Calculating macroeconomic surprises and deviations from the anchor, we take first releases of macro variables as they should correspond closely to information set of forecasters at the moment of declaring their expectations, contrary to the revised values. The sample covers dates from 1981:Q3 to 2020:Q4.

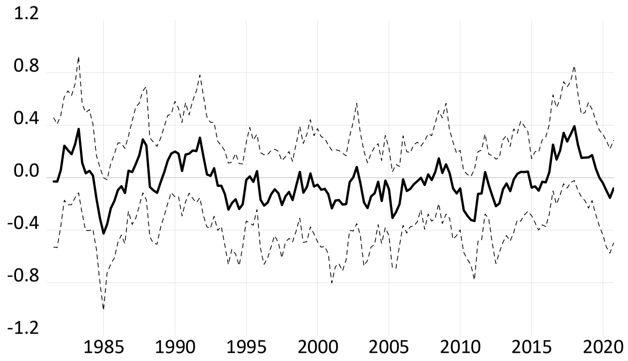
Table 1 shows estimates of main parameters of the models, while Figure 1 and Figure 2 display evolution of estimates of $\beta-$parameters and uncertainty carrier over time. The estimates suggest that GDP growth forecasts are sensitive mainly to GDP growth and interest rate, while inflation forecasts react to all three variables. Estimates of α coefficient confirm that forecasters pay attention to consensus forecast, as put forward in the Assumption 1. The consensus plays slightly larger role for GDP growth forecasts than for inflation forecasts.

Table 1: Parameters estimates

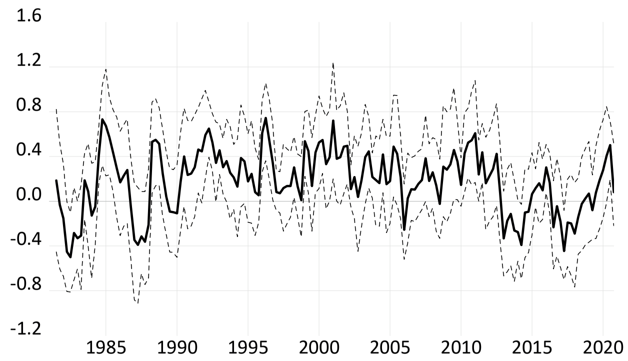
Parameter	Model for GDP growth forecast revisions	Model for inflation forecast revisions
α	0.37 [0.19; 0.54]	0.26 [0.14; 0.39]
μ	-2.44 [-4.13; -1.13]	-4.71 [-6.50; -3.11]
ψ	0.54 [0.30; 0.77]	0.81 [0.59; 0.96]
$E(\beta_{1,t})$	-0.03 [-0.30; 0.22]	0.25 [0.15; 0.37]
$E(\beta_{2,t})$	0.18 [0.00; 0.38]	0.07 [0.01; 0.13]
$E(\beta_{3,t})$	1.85 [0.94; 2.72]	0.54 [0.06; 1.02]
f_2	0.68 [0.43; 0.89]	0.63 [0.44; 0.81]
f_3	0.59 [0.38; 0.80]	0.58 [0.34; 0.78]
f_4	0.67 [0.42; 0.90]	0.63 [0.42; 0.83]

Notes: Table shows posterior means and 90% posterior credible interval (in brackets).

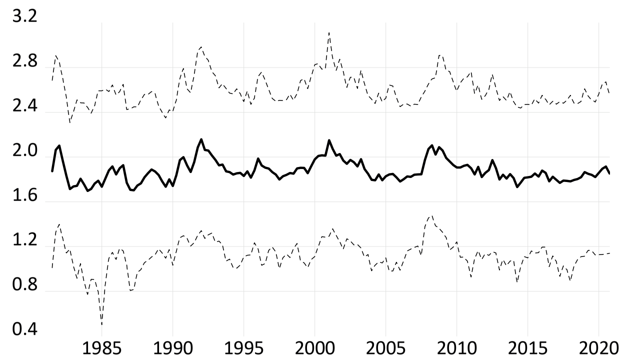
Figure 1: Time varying Betas and the carrier of uncertainty—model for GDP growth forecast revisions



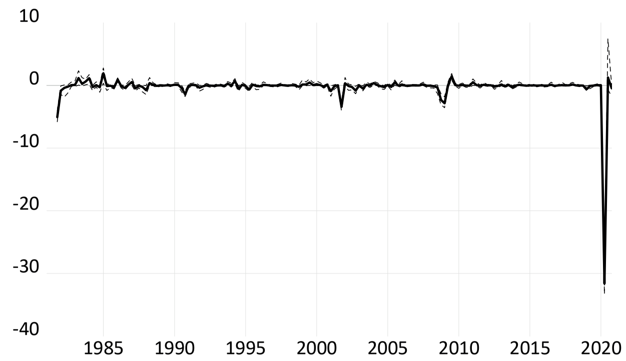
(a) $\beta_{1,t}$



(b) $\beta_{2,t}$



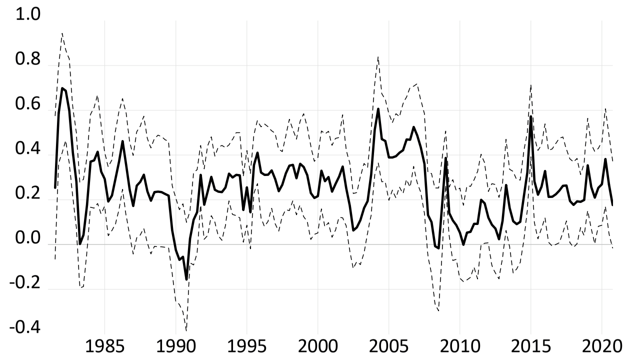
(c) $\beta_{3,t}$



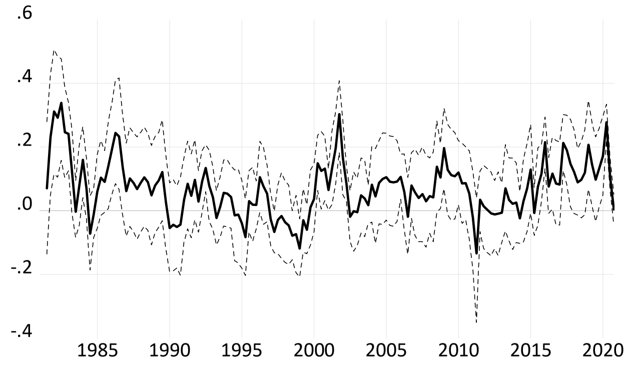
(d) Uncertainty carrier

Notes: Medians (solid line) and 68% posterior error bands (computed point-wise)

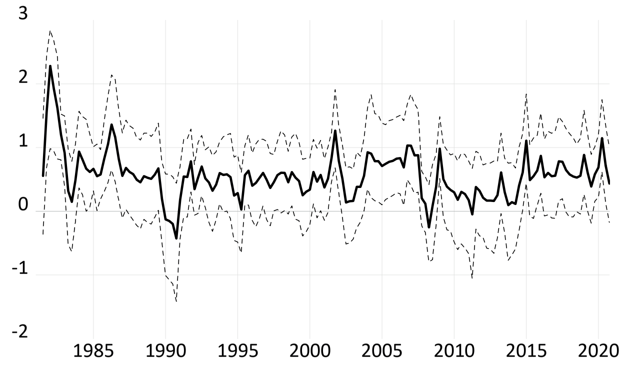
Figure 2: Time varying Betas and the carrier of uncertainty—model for inflation forecast revisions



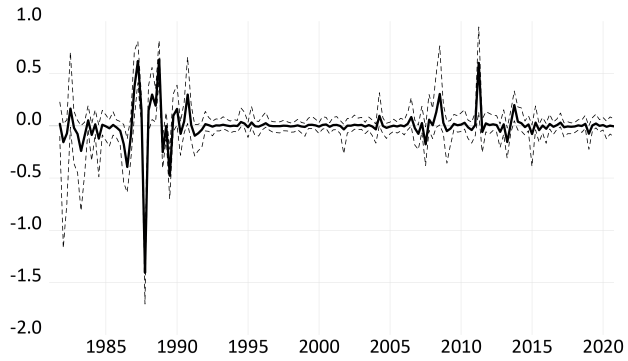
(a) $\beta_{1,t}$



(b) $\beta_{2,t}$



(c) $\beta_{3,t}$



(d) Uncertainty carrier

Notes: Medians (solid line) and 68% posterior error bands (computed point-wise).

Table 2: Statistical properties of uncertainty indices

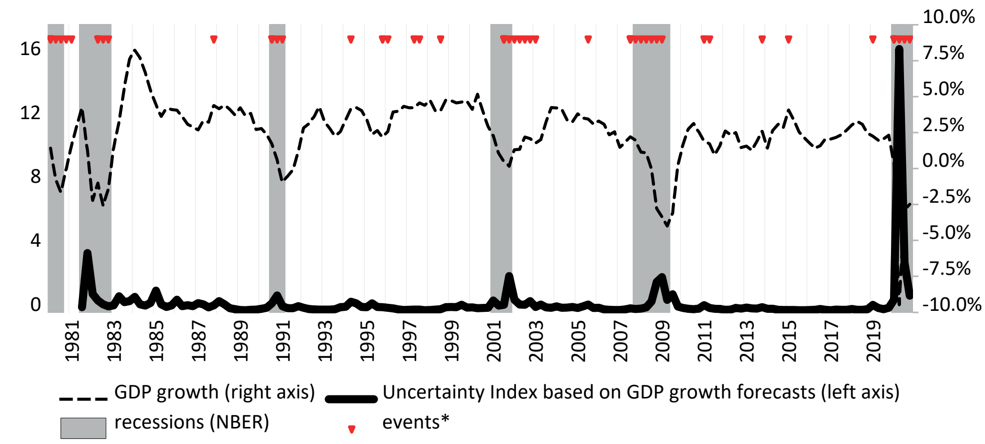
uncertainty index	Persistence$^{(1)}$	Correlation real GDP growth	Correlation CPI inflation
GDP-growth based	0.20***	-0.55***	-0.21***
inflation-based	0.91***	0.02	0.26***

Notes: $^{(1)}$ Parameter in AR(1) model. GDP growth and CPI inflation refer to quarter-to-quarter change. Symbols *** / ** / * denote that correlation is statistically different from zero at the 0.01 / 0.05 / 0.10 levels.

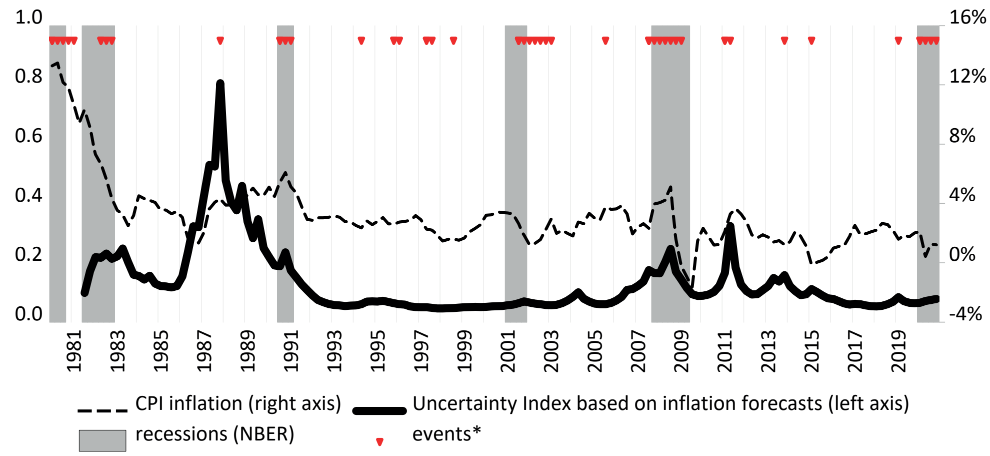
The uncertainty indices based on GDP forecast revisions and inflation forecasts revisions are quite different from each other (Figure 3). The former one is negatively correlated with GDP growth rates[11] and signals uncertainty mainly during recessions (Table 2). We observe enormous increase of this index at the beginning of the Covid-19 pandemic.Uncertainty index based on inflation forecast revisions tends to increase when inflation goes up (Table 2).[12] Its peaks correspond to some important events (Black Monday, Gulf War I, beginning of the global financial crisis, debt ceiling dispute, federal government shut-down) and, especially, to developments related to Fed monetary policy (so-called inflation scares in the 80'ies, Volcker-Greenspan transition, QE programs, uncertainty regarding exit from QE). However, some of the important economic and social events affect this measure of uncertainty only moderately (GFC) or do not affect it at all (beginning of the Covid-19 pandemic). The different evolution of both uncertainty indices during the pandemic is consistent with changes in survey-based measures of forecast disagreement which show a massive increase of disagreement regarding GDP growth forecasts at the beginning of the Covid-19 pandemic and only a mild increase of disagreement regarding inflation forecasts.Given the above findings it seems that both indices describe different kinds of macroeconomic uncertainty. The uncertainty index based on GDP forecast revisions seems to be closely related to developments in economic activity and it reacts mainly to recessions. The uncertainty index based on inflation forecast revisions seems to refer to inflation risks, reflecting in particular uncertainty related to monetary policy frameworks and conduct.

[11]Pearson correlation: -0.63, Spearman correlation: -0.24.
[12]Pearson correlation: 0.39, Spearman correlation: 0.21.

Figure 3: Uncertainty indices and uncertainty events



(a) GDP-growth-based index of uncertainty



(b) Inflation-based index of uncertainty

Notes: * 1987Q4 – Black Monday (19th October), 1990Q3-1991Q1 – Gulf War I, 1994Q2 – surge in commodity prices, 1995Q4-1996Q1 – federal government shutdown, 1997Q2-1997Q3 – Asian Crisis, 1998Q3 – Russian LTCM Default, 2001Q3 – 9/11 terrorist attack, 2001Q4-2002Q3 – Worldcom and Enron, 2002Q4-2003Q1 – Gulf War II, 2005Q3 – hurricane Katrina, 2007Q3-2009Q1 – credit crunch & sub-prime crisis, 2011Q1-2011Q2 – debt ceiling dispute, 2012Q3 – fiscal cliff, 2013Q4 – federal government shutdown, 2015Q1 – coalition against ISIL, 2019Q1 – federal government shutdown, since 2020Q1 – Covid-19 pandemic.

We compare the uncertainty indices with popular uncertainty measures, representing different approaches to capture this unobservable phenomenon. The first one is the volatility of stock market returns measured by VXO index delivered by Chicago Board of Options Exchange. The second one is the Economic Policy Uncertainty (EPUI) by Baker et al. (2016) which is based on text search of leading newspapers for words related to economy, uncertainty and policy. We take under consideration the main EPUI index as well as one of its versions aimed at measuring uncertainty related to monetary policy.[13] Our third benchmark, an index of macroeconomic uncertainty index introduced by Jurado et al. (2015) (thereafter JLN) aggregates signals from large number of economic indicators. Uncertainty in a single indicator is derived from conditional volatility of future series, but only of the part that is unforceastable. Finally, we compare our uncertainty indices with another survey-based measure: disagreement of professional forecasters with regard to future inflation and GDP growth.

The GDP-growth based index of uncertainty correlates with all alternative indices, showing the highest similarity to disagreement in GDP growth forecasts (Table 3). On the contrary, the index based on inflation forecasts correlates only with selected alternative indices. In particular, this index co-moves with the EPUI sub-index related to monetary policy, but not with the baseline EPUI aimed at measuring more general uncertainty. On the contrary, the GDP-growth-based index shows stronger correlation with the general EPUI than with its monetary policy version. This is in line with our interpretation that the two indices capture different types of uncertainty.

[13]Monetary Policy index additionally counts words related to interest rates, the Fed and inflation.

Table 3: Correlation coefficients of the uncertainty indices with alternative uncertainty measures

Uncertainty measure	GDP-growth-forecast based	inflation-forecasts based
VXO	0.29***	0.27***
EPUI	0.65***	0.12
EPUI (Monetary Policy)	0.25***	0.24***
JLN	0.55***	0.15
forecast disagreement (GDP growth)	0.96***	0.10
forecast disagreement (inflation)	0.21***	0.24***

Notes: Symbols *** / ** / * denote that correlation is statistically different from zero at the 0.01 / 0.05 / 0.10 levels. Due to data availability correlations are calculated on different time ranges – VXO: 1986q3 – 2020q4; EPUI: 1985q1 – 2020q4; JLN: 1981q4 - 2011q4; forecast disagreement: 1981q4 – 2020q4. JLN uncertainty measure refers to 3-month-ahead uncertainty. Forecast disagreement is measured by interquartile range of one-quarter-ahead forecasts. In parentheses are shown correlations on sample ending before the Covid-19 pandemic.

Source: own calculations based on data from: CBOB, Baker et al. (2016), Jurado et al. (2015), https://www.philadelphiafed.org/research-and-data/real-time-center/survey-of-professional-forecasters.

5.2 Impact of uncertainty on economy

Finally, we validate the developed uncertainty indices by analyzing the responses to uncertainty shocks obtained from VAR models estimated on quarterly data. Their specification is similar to earlier studies, in particular to Bloom (2009) and Liu and Sheng (2019). However, given than in calculating uncertainty measures we control for macroeconomic surprises, the ordering of uncertainty in the VAR specification is slightly different than in the existing studies. More specifically, we assume that uncertainty does not instantaneously react to shocks to the other variables, but at the same time all the other variables in the system can react to uncertainty shocks within the same quarter. The latter assumption seems relevant empirically given that the consequences of large uncertainty shocks, such as the collapse of Lehman Brothers or the beginning of COVID-19 pandemics, are likely to affect financial markets and macroeconomic variables immediately. The analysis of the uncertainty indices proposed in this study (see section 5.1) suggests that they do not necessarily respond to shocks to individual macroeconomic variables, but are affected either by wide-encompassing macroeconomic developments, i.e. recessions (uncertainty index based on GDP revisions) or by institutional changes, related especially to the conduct of monetary policy (uncertainty index based on CPI inflation revisions). Therefore, the ordering of variables in the VAR models applied in this study seems justified from different perspectives.

The VAR models include seven variables in the following order: the uncertainty index, unemployment rate, real GDP (in logs), wages (in logs), prices (in logs), short-term interest rate[14] and S&P 500 Index (in logs). We detrend all series except the uncertainty measures using the HP filter. In line with the lag length information criteria we use VAR models with 2 lags.[15]

Empirical literature suggests that uncertainty shocks act similarly to typical demand shocks (Leduc and Liu, 2016), i.e. they lead to a reduction in GDP, increase of unemployment and a fall in inflation. However, as Fasani and Rossi (2018) show, under specific monetary policy reaction function, which assumes that monetary authorities react slowly, by smoothing the interest rate, uncertainty shocks can propagate as supply shocks, pushing inflation up. Our results (Figures 4-5) tilt towards the former interpretation. Shocks to both uncertainty indices developed in our study lead to economic contraction, reflected in a fall in real GDP and an increase of unemployment, as well as to a reduction in CPI inflation. Depending on the uncertainty index applied, the impact of its shock on GDP growth lasts 3-4 quarters, the responses of unemployment rate seems to be a bit longer (4-8 quarters), while the response of CPI inflation lasts 1-4 quarters. Comparing the above findings with the previous empirical literature, the responses of economic activity in our study seem more persistent than in Bloom (2009), but comparable to Jurado et al. (2015).

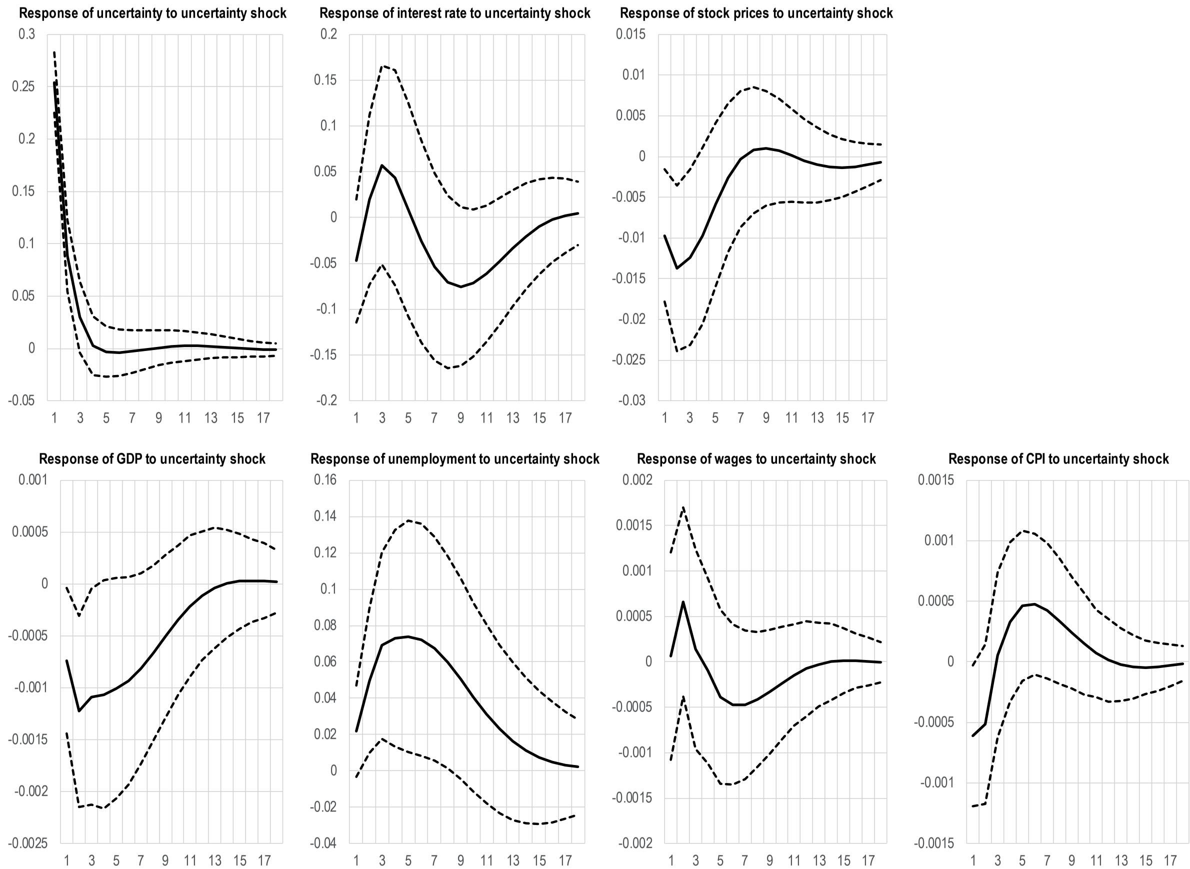
We conducted a similar analysis including in VAR models other uncertainty measures used previously in the literature. We applied the same ordering as in the case of VAR models estimated with both uncertainty proxies developed in this study. The results, presented in Figure 6 (see Appendix A.6 for details), suggest that the impact of uncertainty shocks on GDP growth, unemployment rate and CPI inflation is similar across various uncertainty measures.[16] The

[14]In the benchmark specifications we apply the shadow interest rate derived by Wu and Xia (2016), data source: `https://sites.google.com/site/jingcynthiawu/home/wu-xia-shadow-rates`.

[15]In the robustness analysis we used similar VAR models with 4 lags. In addition, we estimated analogous models based on levels of the variables considered. The qualitative conclusions remain broadly unchanged. The results are available on request from the authors.

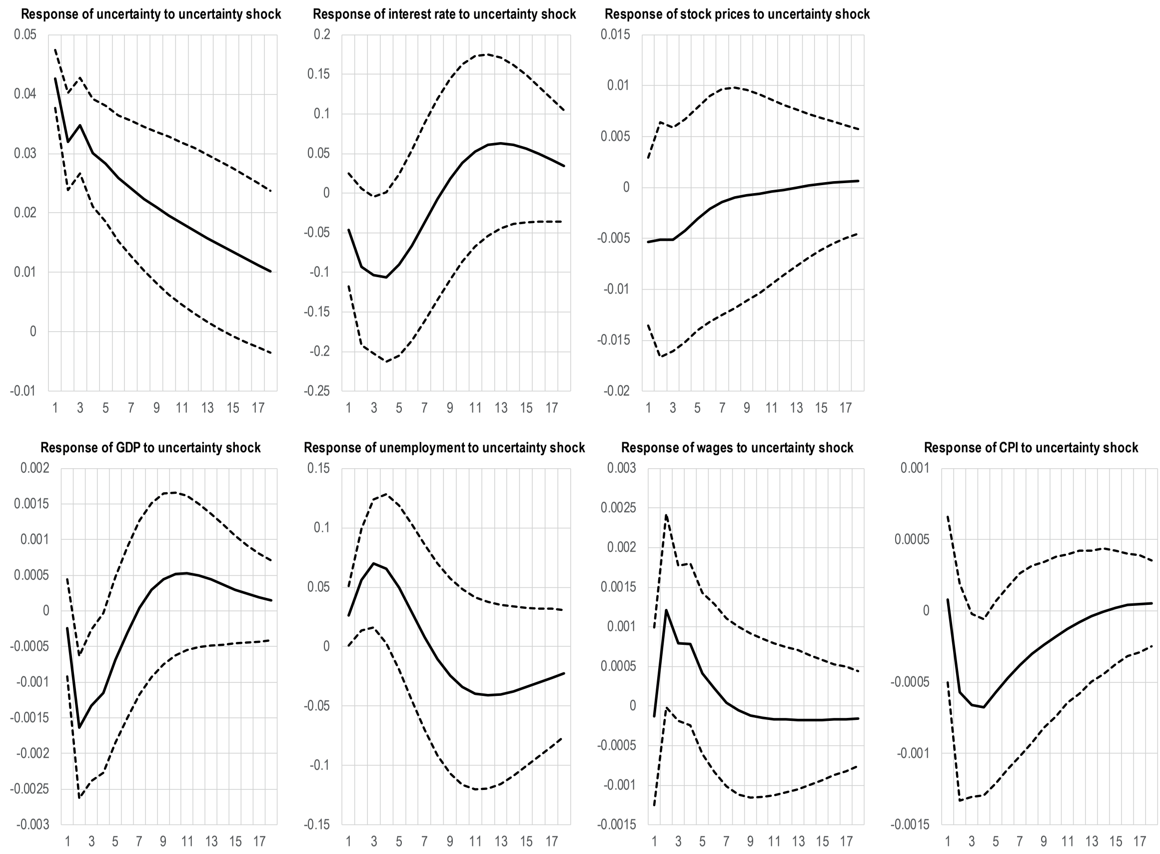
[16]Interestingly, applying the conventional ordering in the VAR models that allows uncertainty to respond instantaneously to shocks of all the other variables in the system and makes macroeconomic variables respond to uncertainty shocks only with a delay, a majority of impulse responses become statistically insignificant. It suggests that allowing for instantaneous impact of uncertainty shocks on macroeconomic developments is useful in analyzing the propagation of uncertainty shocks and leads to more consistent results, which remain robust across different proxies of uncertainty used in the model.

Figure 4: Responses to uncertainty shock (uncertainty index based on GDP growth forecasts)



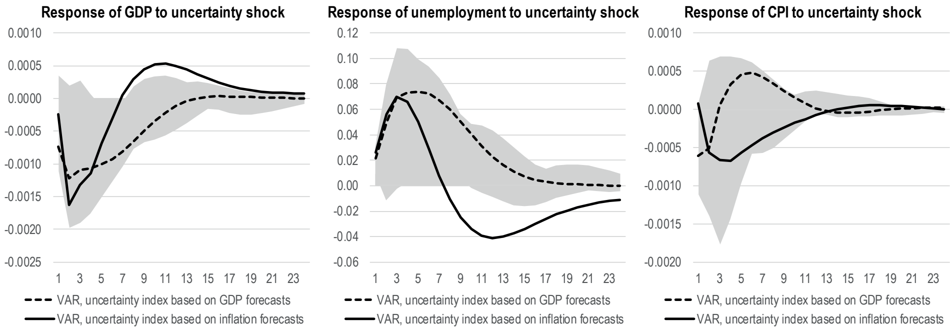
Notes: The dashed lines show 95% confidence interval.

Figure 5: Responses to uncertainty shock (uncertainty index based on inflation forecasts)



Notes: The dashed lines show 95% confidence interval.

Figure 6: Responses to uncertainty shocks (various measures)



Notes: The figures above present impulse responses of selected variables to uncertainty shocks based on VAR models using different proxies for uncertainty. Solid and dashed lines indicate the responses from VAR models using uncertainty indices proposed in this paper, while the gray area indicates the range of impulse responses from VAR models using alternative measures of uncertainty proposed in previous literature.

Source: own calculations based on the results presented in Figures 4-5 above and in the Appendix A.6.

shocks to uncertainty indices proposed in this paper, especially to uncertainty index based on GDP forecast revisions, produce relatively strong and persistent responses of the measures of economic activity.

6 Conclusions

This paper deals with two important and interrelated issues in economics, namely expectations and uncertainty. Our goal was to build the theory of subjective expectations which reflects realistically the forecasting problem faced by typical individual and which, at the same time, has strong theoretical foundations. We hope to reach this objective, inter alia, by deriving our theory from few descriptive axioms and avoiding explicit reference to probability. An important part of the theory is the subjective assessment hypothesis (SAH)—our proposal of the individual's optimal decision rule in the context of the forecasting task. This rule includes the quest for the most accurate prediction and some form of anchoring of expectations.

We illustrate usefulness of our theory of subjective expectations in empirical research by applying it to derive uncertainty from forecast revisions. In our approach, information about uncertainty is included in the part of forecast revision that is not justified in the light of the best opinion under the hypothetical extra knowledge on current state of the economy. The uncertainty index itself is equal to time-varying volatility of this uncertainty carrier.

Finally, we calculate the uncertainty indices for the US economy over long time span, including the Covid-19 pandemic. They are based on forecast revisions of professional forecasters regarding either GDP growth or inflation. We show that our indices have a meaningful interpretation as their peaks correspond to major events in the American economic history. These indices correlate with alternative popular measures of uncertainty, however, they describe different kinds of macroeconomic uncertainty. We also find that the shocks to both indices act similarly as demand shocks, generating economic contractions and pushing inflation down.

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Appendix

A.1 The proof of proposition 1

By Axiom 3, $\mathbf{1} \in \mathcal{S}$. The smallest linear subspace that contains $\mathbf{1}$ is $\mathcal{S}_0$. Let us fix any $\alpha_* \neq 0$ and construct the new $u.q.$ given by $\mathbf{y} = \alpha_* \mathbf{1} \in \mathcal{S}_0$. Suppose there are two different subjective expectations E and $\bar{E}$. It follows that $E(\mathbf{y}) = E(\alpha_* \mathbf{1}) = \alpha_* E(\mathbf{1}) = \alpha_*$ and $\bar{E}(\mathbf{y}) = \bar{E}(\alpha_* \mathbf{1}) = \alpha_* \bar{E}(\mathbf{1}) = \alpha_*$ (since both subjective expectations must obey $E(\mathbf{1}) = \bar{E}(\mathbf{1}) = 1$). It follows that $E(\mathbf{y}) = \bar{E}(\mathbf{y})$, so that both subjective expectations must be the same on $\mathcal{S}_0$ (i.e the contradiction). The final conclusion follows from standard results in functional analysis. To this end recall that any subjective expectation is the positive linear functional. Let $\mathcal{S}$ be any linear subspace such that $\mathcal{S}_0 \subset \mathcal{S}$. We need to show that if there is a subjective expectation $E(\cdot)$ defined on $\mathcal{S}_0$, then there exists a subjective expectation $E^*(\cdot)$ on $\mathcal{S}$, such that $E(\cdot) = E^*(\cdot)$ on $\mathcal{S}_0$. We call $E^*(\cdot)$ an extension of $E(\cdot)$. Existence of extension of linear functionals is the standard result (see e.g. Taylor (1958), p. 40). Existence of extension of positive linear functionals (i.e. subjective expectations) follows by theorem 1, p. 29, in Lax (2002), which proves existence of such an extension provided that $\mathcal{S}_0$ contains some $u.q.$ $\mathbf{x}_0$, such that $\mathbf{x}_0 \geq \mathbf{1}$. Setting $\mathbf{x}_0 = \mathbf{1} \in \mathcal{S}_0$ proves the result.

A.2 The proof of proposition 2

We need to minimize

$$\|\mathbf{y}_{t+1} - \phi\mathbf{1} - \beta_1\mathbf{x}_{1,t} - \cdots - \beta_n\mathbf{x}_{n,t}\|_h^2 + \lambda\,\|\bar{y}_h\mathbf{1} - \phi\mathbf{1} - \beta_1\mathbf{x}_{1,t} - \cdots - \beta_n\mathbf{x}_{n,t}\|_h^2$$

with respect to $\phi, \beta_1, \ldots, \beta_n$, where $\lambda \geq 0$ and $\bar{y}_h \in \mathbb{R}$ are kept fixed. First note that using our axioms we can write

$$\begin{aligned}
&\|\mathbf{y}_{t+1} - \phi\mathbf{1} - \beta_1\mathbf{x}_{1,t} - \cdots - \beta_n\mathbf{x}_{n,t}\|_h^2 = \\
&\quad = \|\mathbf{y}_{t+1} - \beta_1\mathbf{x}_{1,t} - \cdots - \beta_n\mathbf{x}_{n,t} - E_h(\mathbf{y}_{t+1} - \beta_1\mathbf{x}_{1,t} - \cdots - \beta_n\mathbf{x}_{n,t})\mathbf{1}\| + \\
&\qquad + \|\phi\mathbf{1} - E_h(\mathbf{y}_{t+1} - \beta_1\mathbf{x}_{1,t} - \cdots - \beta_n\mathbf{x}_{n,t})\mathbf{1}\|_h^2
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\lambda\,\|\bar{y}_h\mathbf{1} - \phi\mathbf{1} - \beta_1\mathbf{x}_{1,t} - \cdots - \beta_n\mathbf{x}_{n,t}\|_h^2 = \\
&\quad = \lambda\,\|\bar{y}_h\mathbf{1} - \beta_1\mathbf{x}_{1,t} - \cdots - \beta_n\mathbf{x}_{n,t} - E_h(\bar{y}_h\mathbf{1} - \beta_1\mathbf{x}_{1,t} - \cdots - \beta_n\mathbf{x}_{n,t})\mathbf{1}\|_h^2 + \\
&\qquad + \lambda\,\|\phi\mathbf{1} - E_h(\bar{y}_h\mathbf{1} - \beta_1\mathbf{x}_{1,t} - \cdots - \beta_n\mathbf{x}_{n,t})\mathbf{1}\|_h^2
\end{aligned}$$

Collecting the terms involving ϕ we get

$$\begin{aligned}
&\|\phi\mathbf{1} - E_h(\mathbf{y}_{t+1} - \beta_1\mathbf{x}_{1,t} - \cdots - \beta_n\mathbf{x}_{n,t})\mathbf{1}\|_h^2 + \\
&\qquad + \lambda\,\|\phi\mathbf{1} - E_h(\bar{y}_h\mathbf{1} - \beta_1\mathbf{x}_{1,t} - \cdots - \beta_n\mathbf{x}_{n,t})\mathbf{1}\|_h^2 = \\
&\qquad = (1+\lambda)(\phi - \frac{1}{1+\lambda}(E_h(\mathbf{y}_{t+1} - \beta_1\mathbf{x}_{1,t} - \cdots - \beta_n\mathbf{x}_{n,t}) + \\
&\qquad + \lambda E_h(\bar{y}_h\mathbf{1} - \beta_1\mathbf{x}_{1,t} - \cdots - \beta_n\mathbf{x}_{n,t})))^2 + \frac{\lambda}{1+\lambda}(E_h(\mathbf{y}_{t+1}) - \bar{y}_h)^2
\end{aligned}$$

Clearly setting

$$\begin{aligned}
\phi &= \frac{1}{1+\lambda}E_h(\mathbf{y}_{t+1} - \beta_1\mathbf{x}_{1,t} - \cdots - \beta_n\mathbf{x}_{n,t}) + \frac{\lambda}{1+\lambda}E_h(\bar{y}_h\mathbf{1} - \beta_1\mathbf{x}_{1,t} - \cdots - \beta_n\mathbf{x}_{n,t}) = \\
&= \frac{1}{1+\lambda}E_h(\mathbf{y}_{t+1}) + \frac{\lambda}{1+\lambda}\bar{y}_h - \beta_1 E_h(\mathbf{x}_{1,t}) - \cdots - \beta_n E_h(\mathbf{x}_{n,t})
\end{aligned}$$

uniquely minimizes the overall function. Then

$$
\begin{aligned}
&\|\mathbf{y}_{t+1}-\beta_1\mathbf{x}_{1,t}-\cdots-\beta_n\mathbf{x}_{n,t}-E_h(\mathbf{y}_{t+1}-\beta_1\mathbf{x}_{1,t}-\cdots-\beta_n\mathbf{x}_{n,t})\mathbf{1}\|_h^2+\\
&\quad+\lambda\,\|-\beta_1\mathbf{x}_{1,t}-\cdots-\beta_n\mathbf{x}_{n,t}-E_h(-\beta_1\mathbf{x}_{1,t}-\cdots-\beta_n\mathbf{x}_{n,t})\mathbf{1}\|_h^2=\\
&\quad=(1+\lambda)\Big\|\frac{1}{1+\lambda}(\mathbf{y}_{t+1}-E_h(\mathbf{y}_{t+1})\mathbf{1})-\beta_1\mathbf{x}_{1,t}-\cdots-\beta_n\mathbf{x}_{n,t}+\\
&\qquad-E_h(-\beta_1\mathbf{x}_{1,t}-\cdots-\beta_n\mathbf{x}_{n,t})\mathbf{1}\Big\|_h^2+\frac{\lambda}{1+\lambda}\,\|\mathbf{y}_{t+1}-E_h(\mathbf{y}_{t+1})\mathbf{1}\|_h^2
\end{aligned}
$$

So that the minimization with respect to $\beta_1,\ldots,\beta_n$ is equivalent to minimization $\left\|\frac{1}{1+\lambda}(\mathbf{y}_{t+1}-E_h(\mathbf{y}_{t+1})\mathbf{1})-\beta_1(\mathbf{x}_{1,t}-E_h(\mathbf{x}_{1,t})\mathbf{1})-\cdots-\beta_n(\mathbf{x}_{n,t}-E_h(\mathbf{x}_{n,t})\mathbf{1})\right\|_h$

Clearly, necessary and sufficient condition for the minimum is (see e.g. Luenberger (1969), pp. 50-57)

$$
\begin{aligned}
E_h\Big[(&\frac{1}{1+\lambda}(\mathbf{y}_{t+1}-E_h(\mathbf{y}_{t+1})\mathbf{1})-\beta_1(\mathbf{x}_{1,t}-E_h(\mathbf{x}_{1,t})\mathbf{1})-\cdots\\
&\cdots-\beta_n(\mathbf{x}_{n,t}-E_h(\mathbf{x}_{n,t})\mathbf{1}))\cdot(\mathbf{x}_{1,t}-E_h(\mathbf{x}_{1,t})\mathbf{1})\Big]=0
\end{aligned}
$$

$$
\begin{aligned}
E_h\Big[(&\frac{1}{1+\lambda}(\mathbf{y}_{t+1}-E_h(\mathbf{y}_{t+1})\mathbf{1})-\beta_1(\mathbf{x}_{1,t}-E_h(\mathbf{x}_{1,t})\mathbf{1})-\cdots\\
&\cdots-\beta_n(\mathbf{x}_{n,t}-E_h(\mathbf{x}_{n,t})\mathbf{1}))\cdot(\mathbf{x}_{2,t}-E_h(\mathbf{x}_{2,t})\mathbf{1})\Big]=0
\end{aligned}
$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$
\begin{aligned}
E_h\Big[(&\frac{1}{1+\lambda}(\mathbf{y}_{t+1}-E_h(\mathbf{y}_{t+1})\mathbf{1})-\beta_1(\mathbf{x}_{1,t}-E_h(\mathbf{x}_{1,t})\mathbf{1})-\cdots\\
&\cdots-\beta_n(\mathbf{x}_{n,t}-E_h(\mathbf{x}_{n,t})\mathbf{1}))\cdot(\mathbf{x}_{n,t}-E_h(\mathbf{x}_{n,t})\mathbf{1})\Big]=0
\end{aligned}
$$

Equivalently

$$
\begin{bmatrix}
var_h(\mathbf{x}_{1,t}) & cov_h(\mathbf{x}_{1,t},\mathbf{x}_{2,t}) & \ldots & cov_h(\mathbf{x}_{1,t},\mathbf{x}_{n,t})\\
cov_h(\mathbf{x}_{1,t},\mathbf{x}_{2,t}) & var_h(\mathbf{x}_{2,t}) & \ldots & cov_h(\mathbf{x}_{2,t},\mathbf{x}_{n,t})\\
\vdots & \vdots & \ddots & \vdots\\
cov_h(\mathbf{x}_{1,t},\mathbf{x}_{n,t}) & cov_h(\mathbf{x}_{2,t},\mathbf{x}_{n,t}) & \ldots & var_h(\mathbf{x}_{n,t})
\end{bmatrix}
\begin{bmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_n\end{bmatrix}
=\frac{1}{1+\lambda}
\begin{bmatrix}cov_h(\mathbf{y}_{t+1},\mathbf{x}_{1,t})\\ cov_h(\mathbf{y}_{t+1},\mathbf{x}_{2,t})\\ \vdots\\ cov_h(\mathbf{y}_{t+1},\mathbf{x}_{n,t})\end{bmatrix}
\tag{A.1}
$$

where $var_h(\mathbf{x}_{i,t}) := E_h[(\mathbf{x}_{i,t}-E_h(\mathbf{x}_{i,t})\mathbf{1})\cdot(\mathbf{x}_{i,t}-E_h(\mathbf{x}_{i,t})\mathbf{1})]$, $cov_h(\mathbf{x}_{i,t},\mathbf{x}_{j,t}) := E_h[(\mathbf{x}_{i,t}-E_h(\mathbf{x}_{i,t})\mathbf{1})\cdot(\mathbf{x}_{j,t}-E_h(\mathbf{x}_{j,t})\mathbf{1})]$, and $cov_h(\mathbf{y}_{t+1},\mathbf{x}_{i,t})$ is defined analogously. Hence the vector jointly closest to $\mathbf{y}_{t+1}$ and $\bar{y}_h\mathbf{1}$ according to the decision rule

from the perspective at time "h" is given by
$\hat{\mathbf{y}}_{t+1} = (1-\alpha)[E_h(\mathbf{y}_{t+1})\mathbf{1}+\beta_{1,h}(\mathbf{x}_{1,t}-E_h(\mathbf{x}_{1,t})\mathbf{1})+\cdots+\beta_{n,h}(\mathbf{x}_{n,t}-E_h(\mathbf{x}_{n,t})\mathbf{1})]+\alpha\bar{y}_h\mathbf{1}$

where $\alpha = \frac{\lambda}{1+\lambda}$, and $\beta_{1,h}, \ldots, \beta_{n,h}$ is any solution to the system of linear equations (A.1) (in $\beta_1, \ldots, \beta_n$).

A.3 The proof of proposition 3

It follows from the proof of proposition 2, that minimization with respect to ϕ gives $\phi = (1-\alpha)E_h(\mathbf{y}_{t+1}) + \alpha\bar{y}_h - \beta_1 E_h(\mathbf{x}_{1,t}) - \cdots - \beta_n E_h(\mathbf{x}_{n,t})$. Plugging it into the expression for $\hat{\mathbf{y}}_{t+1}$ results in $\hat{\mathbf{y}}_{t+1} = ((1-\alpha)E_h(\mathbf{y}_{t+1}) + \alpha\bar{y}_h)\mathbf{1} + \beta_1(\mathbf{x}_{1,t} - E_h(\mathbf{x}_{1,t})\mathbf{1}) + \cdots + \beta_n(\mathbf{x}_{n,t} - E_h(\mathbf{x}_{n,t})\mathbf{1})$. Let us denote $\bar{\mathbf{x}} = \beta_1(\mathbf{x}_{1,t} - E_h(\mathbf{x}_{1,t})\mathbf{1}) + \cdots + \beta_n(\mathbf{x}_{n,t} - E_h(\mathbf{x}_{n,t})\mathbf{1})$. Clearly $\hat{\mathbf{y}}_{t+1}$ is unique iff $\bar{\mathbf{x}}$ is. To demonstrate the latter, in the proof of proposition 2 we showed that minimization with respect to $\beta_1, \ldots, \beta_n$ is tantamount to finding $\bar{\mathbf{x}}$ such that $\left\| \frac{1}{1+\lambda}(\mathbf{y}_{t+1} - E_h(\mathbf{y}_{t+1})\mathbf{1}) - \bar{\mathbf{x}} \right\|_h$ attains the minimum. Using the reasoning from standard projection theorems, necessary and sufficient condition for the minimum is that $E_h[(\frac{1}{1+\lambda}(\mathbf{y}_{t+1} - E_h(\mathbf{y}_{t+1})\mathbf{1}) - \bar{\mathbf{x}}) \cdot \tilde{\mathbf{x}}] = 0$, where $\tilde{\mathbf{x}}$ is any vector that belongs to the linear span of $\{(\mathbf{x}_{1,t} - E_h(\mathbf{x}_{1,t})\mathbf{1}), \ldots, (\mathbf{x}_{n,t} - E_h(\mathbf{x}_{n,t})\mathbf{1})\}$. Suppose there is $\tilde{\mathbf{x}} \neq \bar{\mathbf{x}}$ that minimizes $\left\| \frac{1}{1+\lambda}(\mathbf{y}_{t+1} - E_h(\mathbf{y}_{t+1})\mathbf{1}) - \tilde{\mathbf{x}} \right\|_h$. It follows $\left\| \frac{1}{1+\lambda}(\mathbf{y}_{t+1} - E_h(\mathbf{y}_{t+1})\mathbf{1}) - \tilde{\mathbf{x}} + \bar{\mathbf{x}} - \bar{\mathbf{x}} \right\|_h^2 = \left\| \frac{1}{1+\lambda}(\mathbf{y}_{t+1} - E_h(\mathbf{y}_{t+1})\mathbf{1}) - \bar{\mathbf{x}} \right\|_h^2 + \|\bar{\mathbf{x}} - \tilde{\mathbf{x}}\|_h^2 \geq \left\| \frac{1}{1+\lambda}(\mathbf{y}_{t+1} - E_h(\mathbf{y}_{t+1})\mathbf{1}) - \bar{\mathbf{x}} \right\|_h^2$, hence $\tilde{\mathbf{x}}$ must be such that $\|\bar{\mathbf{x}} - \tilde{\mathbf{x}}\|_h^2 = 0$. But by the fundamental property of the norm we have $\|\bar{\mathbf{x}} - \tilde{\mathbf{x}}\|_h^2 = 0$ iff $\bar{\mathbf{x}} = \tilde{\mathbf{x}}$ i.e. the contradiction.

A.4 The proof of proposition 4

Let us rewrite the model more compactly as

$$y_t = z_t \gamma_t + e^{\frac{1}{2}h_t}\epsilon_t; \qquad \epsilon_t \sim N(0,1)$$

$$h_t = \mu + \psi(h_{t-1} - \mu) + u_t; \qquad u_t \sim N(0,\sigma^2)$$

$$\gamma_t = \nu + F(\gamma_{t-1} - \nu) + \omega_t; \qquad \omega_t \sim N(0,\Sigma)$$

where $z_t = [1 \vdots x_t - E_t(\hat{\mathbf{x}}) - \alpha(x_t - \bar{x}_t)]$, $\gamma_t = \begin{bmatrix} c_t \\ \beta_t \end{bmatrix}$. Our identification analysis draws on two facts: 1) both the unconditional data moments exist and their efficient estimators could be easily designed in our stationary environment, and 2) population moments are uniformly globally identified. Hence global identification amounts to the problem of 1-1 correspondence between data moments and parameters in our model. By stationarity we have

$$\begin{aligned} y_t &= z_t(\gamma_t - \nu) + z_t\nu + e^{\frac{1}{2}(h_t - \mu)}e^{\frac{1}{2}\mu}\epsilon_t \\ &= z_t(I - FL)^{-1}\omega_t + z_t\nu + e^{\frac{1}{2}(1-\psi L)^{-1}u_t}e^{\frac{1}{2}\mu}\epsilon_t \end{aligned}$$

where L denotes the lag operator. Let us introduce the self-explaining decomposition $\nu = [\nu_c \vdots \nu_\beta']'$. By independence of disturbances, we have $E(y_t) = z_t\nu = [1 \vdots x_t - E_t(\hat{\mathbf{x}}) \vdots \bar{x}_t - x_t][\nu_c \vdots \nu_\beta' \vdots \alpha\nu_\beta']'$, where $E(\cdot)$ in this appendix denotes the unconditional mathematical expectation (which should not be confused with the subjective expectation, being the studying object in this paper). Suppose that two distinct $(\nu_c, \nu_\beta, \alpha\nu_\beta) \neq (\bar{\nu}_c, \bar{\nu}_\beta, \bar{\alpha}\bar{\nu}_\beta)$ result in the same unconditional expectation, then

$$E\begin{bmatrix} y_t \\ y_{t+1} \\ \vdots \\ y_{t+h} \end{bmatrix} = \begin{bmatrix} 1 \vdots x_t - E_t(\hat{\mathbf{x}}) \vdots \bar{x}_t - x_t \\ 1 \vdots x_{t+1} - E_{t+1}(\hat{\mathbf{x}}) \vdots \bar{x}_{t+1} - x_{t+1} \\ \vdots \\ 1 \vdots x_{t+h} - E_{t+h}(\hat{\mathbf{x}}) \vdots \bar{x}_{t+h} - x_{t+h} \end{bmatrix} \begin{bmatrix} \nu_c \\ \nu_\beta \\ \alpha\nu_\beta \end{bmatrix} \qquad \text{(A.2)}$$

$$= \underbrace{\begin{bmatrix} 1 \vdots x_t - E_t(\hat{\mathbf{x}}) \vdots \bar{x}_t - x_t \\ 1 \vdots x_{t+1} - E_{t+1}(\hat{\mathbf{x}}) \vdots \bar{x}_{t+1} - x_{t+1} \\ \vdots \\ 1 \vdots x_{t+h} - E_{t+h}(\hat{\mathbf{x}}) \vdots \bar{x}_{t+h} - x_{t+h} \end{bmatrix}}_{V} \begin{bmatrix} \bar{\nu}_c \\ \bar{\nu}_\beta \\ \bar{\alpha}\bar{\nu}_\beta \end{bmatrix}$$

Assume that at least one of entries in the vector ν_β is not zero. Provided that V has full column rank (necessary condition is that we have at least $2dim(\beta_t) + 1$ observations) then (A.2) implies $(\nu_c, \nu_\beta, \alpha\nu_\beta) = (\bar{\nu}_c, \bar{\nu}_\beta, \bar{\alpha}\bar{\nu}_\beta)$ i.e. ν_c, ν_β, α are globally identified. This allows us to treat z_t as the given regressors. Let us derive unconditional second moments

$$E\left\{\left(\begin{bmatrix} y_t \\ y_{t+1} \\ \vdots \\ y_{t+h} \end{bmatrix} - E\begin{bmatrix} y_t \\ y_{t+1} \\ \vdots \\ y_{t+h} \end{bmatrix}\right)\left(\begin{bmatrix} y_t \\ y_{t+1} \\ \vdots \\ y_{t+h} \end{bmatrix} - E\begin{bmatrix} y_t \\ y_{t+1} \\ \vdots \\ y_{t+h} \end{bmatrix}\right)'\right\} = \tag{A.3}$$

$$E\begin{pmatrix} z_t(I-FL)^{-1}\omega_t + e^{\frac{1}{2}(1-\psi L)^{-1}u_t} e^{\frac{1}{2}\mu}\epsilon_t \\ z_{t+1}(I-FL)^{-1}\omega_{t+1} + e^{\frac{1}{2}(1-\psi L)^{-1}u_{t+1}} e^{\frac{1}{2}\mu}\epsilon_{t+1} \\ \vdots \\ z_{t+h}(I-FL)^{-1}\omega_{t+h} + e^{\frac{1}{2}(1-\psi L)^{-1}u_{t+h}} e^{\frac{1}{2}\mu}\epsilon_{t+h} \end{pmatrix}$$

$$\cdot \begin{pmatrix} z_t(I-FL)^{-1}\omega_t + e^{\frac{1}{2}(1-\psi L)^{-1}u_t} e^{\frac{1}{2}\mu}\epsilon_t \\ z_{t+1}(I-FL)^{-1}\omega_{t+1} + e^{\frac{1}{2}(1-\psi L)^{-1}u_{t+1}} e^{\frac{1}{2}\mu}\epsilon_{t+1} \\ \vdots \\ z_{t+h}(I-FL)^{-1}\omega_{t+h} + e^{\frac{1}{2}(1-\psi L)^{-1}u_{t+h}} e^{\frac{1}{2}\mu}\epsilon_{t+h} \end{pmatrix}'$$

in particular we get

$$
\begin{aligned}
var(y_t) =& E(z_t(I-FL)^{-1}\omega_t + e^{\frac{1}{2}(1-\psi L)^{-1}u_t} e^{\frac{1}{2}\mu}\epsilon_t)^2 \\
=& z_t(\sum_{i=0}^{\infty} F^i \Sigma F'^i) z_t' + e^{\mu} E(e^{(1-\psi L)^{-1}u_t}) \\
=& z_t(\sum_{i=0}^{\infty} F^i \Sigma F'^i) z_t' + e^{\mu} E(e^{u_t+\psi u_{t-1}+\psi^2 u_{t-2}+\dots}) \\
=& z_t(\sum_{i=0}^{\infty} F^i \Sigma F'^i) z_t' + e^{\mu} \prod_{i=0}^{\infty} E(e^{\psi^i u_{t-i}}) = z_t(\sum_{i=0}^{\infty} F^i \Sigma F'^i) z_t' + e^{\mu} \prod_{i=0}^{\infty} e^{\frac{1}{2}(\psi^i)^2\sigma^2} \\
=& z_t(\sum_{i=0}^{\infty} F^i \Sigma F'^i) z_t' + e^{\mu} e^{\frac{1}{2}\sigma^2/(1-\psi^2)}
\end{aligned} \tag{A.4}
$$

Noting that $\sum_{i=0}^{\infty} F^i \Sigma F'^i$ is just the unconditional covariance of γ_t i.e. $cov(\gamma_t) := E(\gamma_t - \nu)(\gamma_t - \nu)'$, we have $\sum_{i=0}^{\infty} F^i \Sigma F'^i = F cov(\gamma_t) F' + \Sigma$. Further it is easy to see $cov(y_{t+k}, y_t) = z_{t+k} F^k \cdot cov(\gamma_t) z_t'$, $cov(y_{t+2}, y_{t+1}) = z_{t+2} F \cdot cov(\gamma_t) z_{t+1}'$ etc. Let us denote $P = cov(\gamma_t)$ and $q = e^{\mu} e^{\frac{1}{2}\sigma^2/(1-\psi^2)}$, then the whole data covariance (A.3) reads (up to sufficient details)

$$
\begin{bmatrix}
z_t P z_t' + q & z_t P F' z_{t+1}' & z_t P F'^2 z_{t+2}' & & \dots \\
z_{t+1} F P z_t' & z_{t+1} P z_{t+1}' + q & & & \dots \\
z_{t+2} F^2 P z_t' & z_{t+2} F P z_{t+1}' & z_{t+2} P z_{t+2}' + q & & \\
z_{t+3} F^3 P z_t' & z_{t+3} F^2 P z_{t+1}' & z_{t+3} F P z_{t+2}' & z_{t+3} P z_{t+3}' + q & \\
z_{t+4} F^4 P z_t' & z_{t+4} F^3 P z_{t+1}' & z_{t+4} F^2 P z_{t+2}' & z_{t+4} F P z_{t+3}' & \dots \\
\vdots & & & & \ddots \\
z_{t+h} F^h P z_t' & & & &
\end{bmatrix} \tag{A.5}
$$

Consider the subdiagonal elements i.e. $z_{t+i+1} F P z_{t+i}'$ for $i = 0, 1, \dots$. Suppose that two distinct $(F, P) \neq (\bar{F}, \bar{P})$ result in the same covariances then

$$
\begin{bmatrix}
z_t \otimes z_{t+1} \\
z_{t+1} \otimes z_{t+2} \\
z_{t+2} \otimes z_{t+3} \\
z_{t+3} \otimes z_{t+4} \\
\vdots
\end{bmatrix} (vec(FP) - vec(\bar{F}\bar{P})) = 0 \tag{A.6}
$$

if the first matrix on the left has full column rank then $FP = \bar{F}\bar{P}$. On the other hand let us consider elements $z_{t+i+2}F^2Pz'_{t+i}$ for $i = 0, 1, ...$

$$\begin{bmatrix} z_t \otimes z_{t+2} \\ z_{t+1} \otimes z_{t+3} \\ z_{t+2} \otimes z_{t+4} \\ z_{t+3} \otimes z_{t+5} \\ \vdots \end{bmatrix} (vec(F^2P) - vec(\bar{F}^2\bar{P})) = 0 \tag{A.7}$$

if the first matrix on the left has full column rank then $F^2P = \bar{F}^2\bar{P}$. Taking these together we have $F^2P = \bar{F}^2\bar{P} = \bar{F}\bar{F}\bar{P} = \bar{F}FP$. Since Σ is positive definite, P is positive definite too (hence nonsingular). Assuming that F is nonsingular we get $F^2 = \bar{F}F \Leftrightarrow F = \bar{F}$ i.e. F is globally identified. Of course this implies that $P = \bar{P}$. Since $P = FPF' + \Sigma = \bar{F}\bar{P}\bar{F}' + \bar{\Sigma}$ it follows $\Sigma = \bar{\Sigma}$. Keeping this in mind from the expression for $var(y_t)$ we also have $e^{\mu}e^{\frac{1}{2}\sigma^2/(1-\psi^2)} = e^{\bar{\mu}}e^{\frac{1}{2}\bar{\sigma}^2/(1-\bar{\psi}^2)}$. This is all we can get concerning parameters identification using the second moments. It can be shown that the third moments are identically equal to zero hence they are useless from identification point of view. Let us derive the fourth moments. In particular

$$\begin{aligned} E(y_t - E(y_t))^4 =& E(z_t(I - FL)^{-1}\omega_t + e^{\frac{1}{2}(1-\psi L)^{-1}u_t}e^{\frac{1}{2}\mu}\epsilon_t)^4 = E(z_t(I - FL)^{-1}\omega_t)^4 \\ &+6E(z_t(I - FL)^{-1}\omega_t)^2E(e^{\frac{1}{2}(1-\psi L)^{-1}u_t}e^{\frac{1}{2}\mu}\epsilon_t)^2 \\ &+E(e^{\frac{1}{2}(1-\psi L)^{-1}u_t}e^{\frac{1}{2}\mu}\epsilon_t)^4 = E(z_t(I - FL)^{-1}\omega_t)^4 \\ &+6z_tPz'_te^{\mu}E(e^{(1-\psi L)^{-1}u_t}) + 3e^{2\mu}E(e^{2(1-\psi L)^{-1}u_t}) \\ =& E(z_t(I - FL)^{-1}\omega_t)^4 + 6z_tPz'_te^{\mu}e^{\frac{1}{2}\sigma^2/(1-\psi^2)} + 3e^{2\mu}e^{2\sigma^2/(1-\psi^2)} \end{aligned} \tag{A.8}$$

This implies that $e^{2\mu}e^{2\sigma^2/(1-\psi^2)} = e^{2\bar{\mu}}e^{2\bar{\sigma}^2/(1-\bar{\psi}^2)}$. Since we concluded earlier that $e^{\mu}e^{\frac{1}{2}\sigma^2/(1-\psi^2)} = e^{\bar{\mu}}e^{\frac{1}{2}\bar{\sigma}^2/(1-\bar{\psi}^2)}$, we arrive at the conclusion that unconditional moments of h_t are globally identified. Further

$$
\begin{aligned}
E(y_t - E(y_t))^2(y_{t+1} - E(y_{t+1}))^2 = {} & E[(z_t(I-FL)^{-1}\omega_t)^2 + \\
& + 2(z_t(I-FL)^{-1}\omega_t)e^{\frac{1}{2}(1-\psi L)^{-1}u_t}e^{\frac{1}{2}\mu}\epsilon_t + e^{(1-\psi L)^{-1}u_t}e^{\mu}\epsilon_t^2)] \\
& \cdot [(z_{t+1}(I-FL)^{-1}\omega_{t+1})^2 + 2(z_{t+1}(I-FL)^{-1}\omega_{t+1})e^{\frac{1}{2}(1-\psi L)^{-1}u_{t+1}}e^{\frac{1}{2}\mu}\epsilon_{t+1} + \\
& + e^{(1-\psi L)^{-1}u_{t+1}}e^{\mu}\epsilon_{t+1}^2)] = E[(z_t(I-FL)^{-1}\omega_t)^2(z_{t+1}(I-FL)^{-1}\omega_{t+1})^2] + \\
& + E[(z_t(I-FL)^{-1}\omega_t)^2]e^{\mu}E(e^{(1-\psi L)^{-1}u_{t+1}}) + E[(z_{t+1}(I-FL)^{-1}\omega_{t+1})^2 e^{\mu} \\
& \cdot E(e^{(1-\psi L)^{-1}u_t}) + e^{2\mu}E(e^{(1-\psi L)^{-1}u_t + (1-\psi L)^{-1}u_{t+1}}) = \\
& = E[(z_t(I-FL)^{-1}\omega_t)^2(z_{t+1}(I-FL)^{-1}\omega_{t+1})^2] + z_t P z_t' e^{\mu} e^{\frac{1}{2}\sigma^2/(1-\psi^2)} + \\
& + z_{t+1} P z_{t+1}' e^{\mu} e^{\frac{1}{2}\sigma^2/(1-\psi^2)} + e^{2\mu} e^{\frac{1}{2}\sigma^2} e^{\frac{1}{2}(1+\psi)^2\sigma^2/(1-\psi^2)}
\end{aligned}
\tag{A.9}
$$

Since unconditional moments of h_t are identified, this implies

$$
e^{\frac{1}{2}\sigma^2} e^{\frac{1}{2}(1+\psi)^2\sigma^2/(1-\psi^2)} = e^{\frac{1}{2}\bar{\sigma}^2} e^{\frac{1}{2}(1+\bar{\psi})^2\bar{\sigma}^2/(1-\bar{\psi}^2)}
$$

which results in the equation $\frac{\sigma^2}{1-\psi} = \frac{\bar{\sigma}^2}{1-\bar{\psi}}$ (we used the fact that $|\psi| < 1$). But we showed that $\frac{\sigma^2}{1-\psi^2} = \frac{\bar{\sigma}^2}{1-\bar{\psi}^2}$ iff $\frac{1}{1+\psi}\frac{\sigma^2}{1-\psi} = \frac{1}{1+\bar{\psi}}\frac{\bar{\sigma}^2}{1-\bar{\psi}}$ iff $\frac{1}{1+\psi} = \frac{1}{1+\bar{\psi}}$ iff $\psi = \bar{\psi}$. This implies $\sigma^2 = \bar{\sigma}^2$. Hence we showed that under the specified conditions the model is globally identified.

A.5 Sampling

In this appendix we provide the details of the Bayesian inference which is the Gibbs sampling. Hence it is sufficient to derive the so-called full conditional distributions i.e. conditional posterior of the specific (block of) parameters given all the remaining ones. In line with the Bayesian literature we augment the parameter set with latent processes $\{h_t\}$, $\{\gamma_t\}$, so that they are treated on the same footing as the original model parameters. We also assume that the prior decomposes as $p(\alpha,\mu,\psi,\sigma^2,\nu,F,\Sigma) = p(\alpha)p(\mu)p(\psi)p(\sigma^2)p(\nu)p(F)p(\Sigma)$. Lastly, in what follows, we use the traditional abbreviation for Normal (Gaussian) distribution with mean a and covariance b i.e. $N(a,b)$, and to save the space we will denote $\sum := \sum_{t=1}^{T}$

A.5.1 Full conditional posterior of α

... is proportional to

$$p(\alpha)\cdot\prod_{t=1}^{T} exp\{-\frac{1}{2}\frac{(y_t-c_t-(x_t-E_t(\hat{\mathbf{x}}))\beta_t-\alpha(\bar{x}_t-x_t)\beta_t)^2}{e^{h_t}}\}$$
$$= p(\alpha)\cdot\prod_{t=1}^{T} exp\{-\frac{1}{2}\frac{(\alpha-(y_t-c_t-(x_t-E_t(\hat{\mathbf{x}}))\beta_t)/(\bar{x}_t-x_t)\beta_t)^2}{e^{h_t}/((\bar{x}_t-x_t)\beta_t)^2}\}$$

Let us denote $\delta_t = (y_t-c_t-(x_t-E_t(\hat{\mathbf{x}}))\beta_t)/(\bar{x}_t-x_t)\beta_t$ and $\sigma_t^2 = e^{h_t}/((\bar{x}_t-x_t)\beta_t)^2$, then

$$= p(\alpha)\cdot exp\{-\frac{1}{2}(\alpha^2\sum\frac{1}{\sigma_t^2}-2\alpha\sum\frac{\delta_t}{\sigma_t^2}+\sum\frac{\delta_t^2}{\sigma_t^2})\}$$
$$= p(\alpha)\cdot exp\{-\frac{1}{2}(\sum\frac{1}{\sigma_t^2})[\alpha^2-2\alpha(\sum\frac{\delta_t}{\sigma_t^2})(\sum\frac{1}{\sigma_t^2})^{-1}+(\sum\frac{\delta_t^2}{\sigma_t^2})(\sum\frac{1}{\sigma_t^2})^{-1}]\}$$

It follows that full conditional posterior of α is proportional to

$$p(\alpha)\cdot N((\sum\frac{\delta_t}{\sigma_t^2})(\sum\frac{1}{\sigma_t^2})^{-1},(\sum\frac{1}{\sigma_t^2})^{-1})$$

with the support restricted to $(0,1)$. In application we assume $p(\alpha)$ is Gaussian on $(0,1)$, so that the full conditional posterior is also (truncated) Gaussian.

A.5.2 Full conditional posterior of σ^2

... is proportional to

$$p(\sigma^2) \cdot \frac{1}{\sigma} exp\{-\frac{1}{2\underline{f}_1 \sigma^2}(h_0 - \mu)^2\} \prod_{t=1}^{T} \frac{1}{\sigma} exp\{-\frac{1}{2\sigma^2}(h_t - \mu - \psi(h_{t-1} - \mu))^2\}$$
$$= p(\sigma^2) \cdot \frac{1}{\sigma^{T+1}} exp\{-\frac{1}{2\sigma^2}[(h_0 - \mu)^2 \underline{f}_1^{-1} + \sum(h_t - \mu - \psi(h_{t-1} - \mu))^2]\}$$

In application we assume $p(\sigma^2)$ is the density of the inverted gamma distribution so that the conditional posterior of σ^2 is also inverted gamma distribution.

A.5.3 Full conditional posterior of ψ

... is proportional to

$$p(\psi) \cdot \prod_{t=1}^{T} exp\{-\frac{1}{2\sigma^2}(h_t - \mu - \psi(h_{t-1} - \mu))^2\}$$
$$\propto p(\psi) \cdot N(\bar{\psi}, \bar{\sigma}_\psi^2)$$

where $\bar{\psi} = (\sum(h_t - \mu)(h_{t-1} - \mu)) / \sum(h_{t-1} - \mu)^2$ and $\bar{\sigma}_\psi^2 = \sigma^2 / \sum(h_{t-1} - \mu)^2$, with the support restricted to $(-1, 1)$. In application we assume $p(\psi)$ is Gaussian on $(-1, 1)$, hence the full conditional posterior is also (truncated) Gaussian.

A.5.4 Full conditional posterior of μ

... is proportional to

$$p(\mu) \cdot exp\{-\frac{1}{2\sigma^2}[(h_0 - \mu)^2 \underline{f}_1^{-1} + \sum(h_t - \mu - \psi(h_{t-1} - \mu))^2]\}$$
$$\propto p(\mu) \cdot N(\bar{\mu}, \bar{\sigma}_\mu^2)$$

where $\bar{\mu} = (h_0 \underline{f}_1^{-1} + (1 - \psi) \sum(h_t - \psi h_{t-1})) / (\underline{f}_1^{-1} + T(1 - \psi)^2)$ and $\bar{\sigma}_\mu^2 = \sigma^2 / (\underline{f}_1^{-1} + T(1 - \psi)^2)$. In application we assume $p(\mu)$ is the flat prior.

A.5.5 Full conditional posterior of Σ

... is proportional to

$$p(\Sigma)\cdot\prod_{t=1}^{T}|\Sigma|^{-\frac{1}{2}}\,exp\{-\frac{1}{2}(\gamma_t-(I-F)\nu-F\gamma_{t-1})'\Sigma^{-1}(\gamma_t-(I-F)\nu-F\gamma_{t-1})\}\times$$

$$\times\,|\Sigma|^{-\frac{1}{2}}\,exp\{-\frac{1}{2}(\gamma_0-\nu)'\underline{f}_2^{-1}\Sigma^{-1}(\gamma_0-\nu)\}$$

$$\propto p(\Sigma)\cdot|\Sigma|^{-\frac{1}{2}(T+1)}\cdot$$

$$\cdot exp\{-\frac{1}{2}tr\{\Sigma^{-1}(\sum_{t=1}^{T}(\gamma_t-(I-F)\nu-F\gamma_{t-1})(\gamma_t-(I-F)\nu-F\gamma_{t-1})'+\underline{f}_2^{-1}(\gamma_0-\nu)(\gamma_0-\nu)')\}\}$$

In application we assume $p(\Sigma)$ is the density of the inverted Wishart distribution so that the conditional posterior of Σ is also inverted Wishart distribution.

A.5.6 Full conditional posterior of $h_1,...,h_T$

We draw from the underlying distribution using the idea put forward in Kim et al. (1998). To this end we approximate the density of $ln(\epsilon_t^2)$ by the mixture of seven Gaussian densities with means $m_i-1,2704$ and variances υ_i^2, hence the joint density of $h_1,...,h_T$ will be proportional to

$$\prod_{t=1}^{T}\sum_{i=1}^{7}Pr(S_t=i)\frac{1}{\sqrt{2\pi}\upsilon_i}exp\{-\frac{(y_t^\star-h_t-m_i+1,2704)^2}{2\upsilon_i^2}\}\times$$

$$\times\prod_{t=1}^{T}exp\{-\frac{1}{2\sigma^2}(h_t-\mu-\psi(h_{t-1}-\mu))^2\}$$

where $y_t^\star=2ln(y_t-z_t\gamma_t)$ and $\{S_1,S_2,\ldots,S_T\}$ are indicator variables (which become auxilary variables to be drawn). See Kim et al. (1998) for the specification of constants $\{Pr(S_t=i),m_i,\upsilon_i^2\}$, for $i=1,\ldots,7$ and further sampling details.

A.5.7 Full conditional posterior of $\gamma_1,...,\gamma_T$

... is proportional to

$$\prod_{t=1}^{T}exp\{-\frac{1}{2}(\gamma_t-(I-F)\nu-F\gamma_{t-1})'\Sigma^{-1}(\gamma_t-(I-F)\nu-F\gamma_{t-1})\}\times$$

$$\times \prod_{t=1}^{T} e^{-\frac{1}{2}h_t} exp\{-\frac{1}{2}\frac{(y_t - z_t\gamma_t)^2}{e^{h_t}}\}$$

hence it conforms to the standard state-space model. To draw the states $\gamma_1, ..., \gamma_T$, we apply classical multi-move algortithms, see e.g. Frühwirth-Schnatter (1994).

A.5.8 Full conditional posterior of F

... is proportional to

$$p(F) \cdot \prod_{t=1}^{T} exp\{-\frac{1}{2}(\gamma_t - (I-F)\nu - F\gamma_{t-1})'\Sigma^{-1}(\gamma_t - (I-F)\nu - F\gamma_{t-1})\}$$

let us denote $Y = [\gamma_1 - \nu \vdots \gamma_2 - \nu \vdots \dots \vdots \gamma_T - \nu]$, $X = [\gamma_0 - \nu \vdots \gamma_1 - \nu \vdots \dots \vdots \gamma_{T-1} - \nu]$, $K = dim(\gamma_i)$. Note that

$$\prod_{t=1}^{T} exp\{-\frac{1}{2}(\gamma_t - (I-F)\nu - F\gamma_{t-1})'\Sigma^{-1}(\gamma_t - (I-F)\nu - F\gamma_{t-1})\}$$

$$= exp\{tr\{-\frac{1}{2}(Y - FX)'\Sigma^{-1}(Y - FX)\}\}$$

$$= exp\{-\frac{1}{2}(vec(Y) - vec(FX))'(I_T \otimes \Sigma^{-1})(vec(Y) - vec(FX))\}$$

$$= exp\{-\frac{1}{2}(vec(Y) - (X' \otimes I_K)vec(F))'(I_T \otimes \Sigma^{-1})(vec(Y) - (X' \otimes I_K)vec(F))\}$$

writing $vec(F) = \Upsilon f$, where Υ is a known (selection) matrix and f is the $K \times 1$ vector comprising diagonal elements of F (we assumed that the latter is diagonal matrix), the full conditional posterior is proportional to

$$p(f) \cdot exp\{-\frac{1}{2}(f - \bar{f})'\Upsilon'(X \otimes I_K)(I_T \otimes \Sigma^{-1})(X' \otimes I_K)\Upsilon(f - \bar{f})\}$$

$$= p(f) \cdot exp\{-\frac{1}{2}(f - \bar{f})'\Upsilon'(XX' \otimes \Sigma^{-1})\Upsilon(f - \bar{f})\} \propto p(f) \cdot N(\bar{f}, \bar{\Sigma}_f)$$

where $\bar{f} = (\Upsilon'(XX' \otimes \Sigma^{-1})\Upsilon)^{-1}\Upsilon'(X \otimes \Sigma^{-1})vec(Y)$ and $\bar{\Sigma}_f = (\Upsilon'(XX' \otimes \Sigma^{-1})\Upsilon)^{-1}$. The support is restricted so as $f_i \in (-1, 1)$ for all $i = 1, ..., K$. In application we assume $p(f)$ is Gaussian, hence the full conditional posterior is also (truncated) Gaussian.

A.5.9 Full posterior of ν

... is proportional to

$$p(\nu) \cdot exp\{-\frac{1}{2}(\gamma_0 - \nu)' \underline{f}_2^{-1} \Sigma^{-1} (\gamma_0 - \nu)\} \cdot$$
$$\cdot \prod_{t=1}^{T} exp\{-\frac{1}{2}(\gamma_t - (I-F)\nu - F\gamma_{t-1})' \Sigma^{-1} (\gamma_t - (I-F)\nu - F\gamma_{t-1})\}$$

$$\propto p(\nu) \cdot N(\bar{\nu}, \bar{\Sigma}_\nu)$$

where $\bar{\nu} = (T(I - F')\Sigma^{-1}(I - F) + \underline{f}_2^{-1}\Sigma^{-1})^{-1}((I - F')\Sigma^{-1} \sum(\gamma_t - F\gamma_{t-1}) + \underline{f}_2^{-1}\Sigma^{-1}\gamma_0)$ and $\bar{\Sigma}_\nu = (T(I - F')\Sigma^{-1}(I - F) + \underline{f}_2^{-1}\Sigma^{-1})^{-1}$. In application we assume $p(\nu)$ is the flat prior.

A.6 Impact of other uncertainty measures on economy

Figure A.1: Responses to uncertainty shock (VXO uncertainty index)

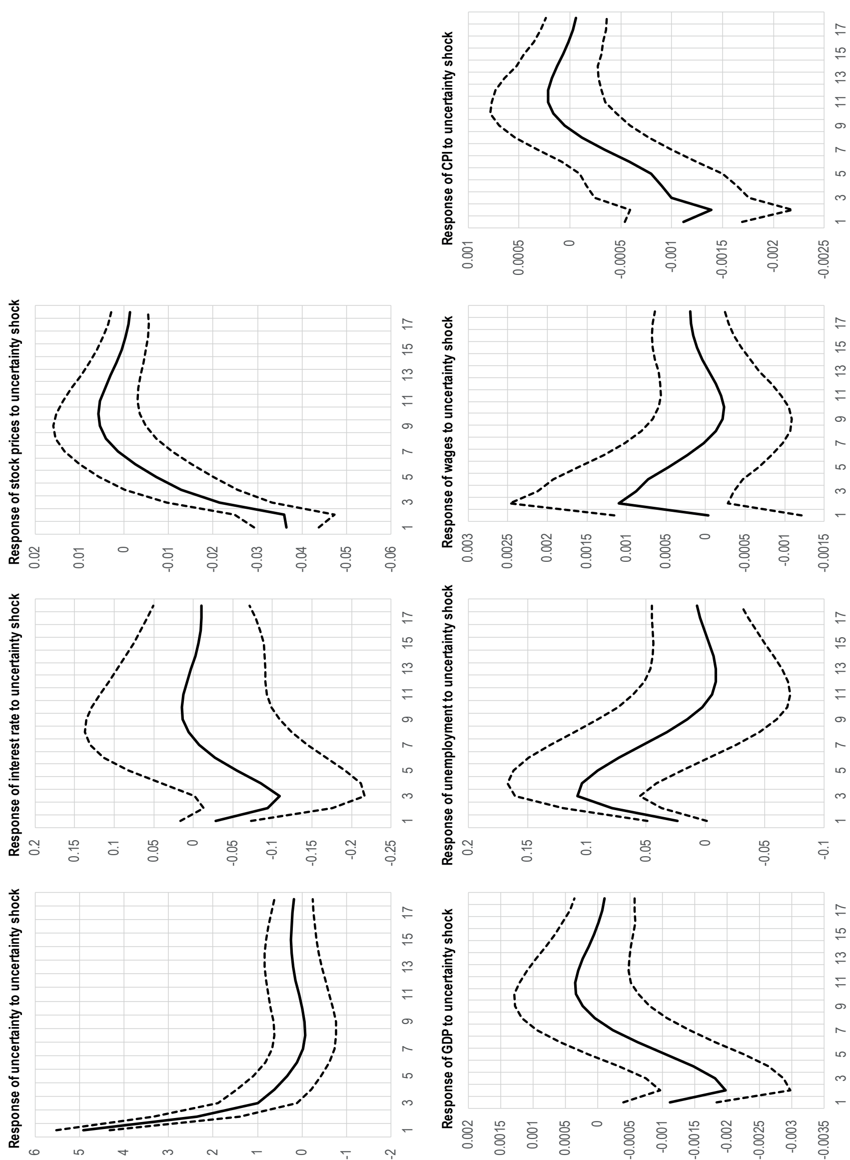


Figure A.2: Responses to uncertainty shock (EPUI)

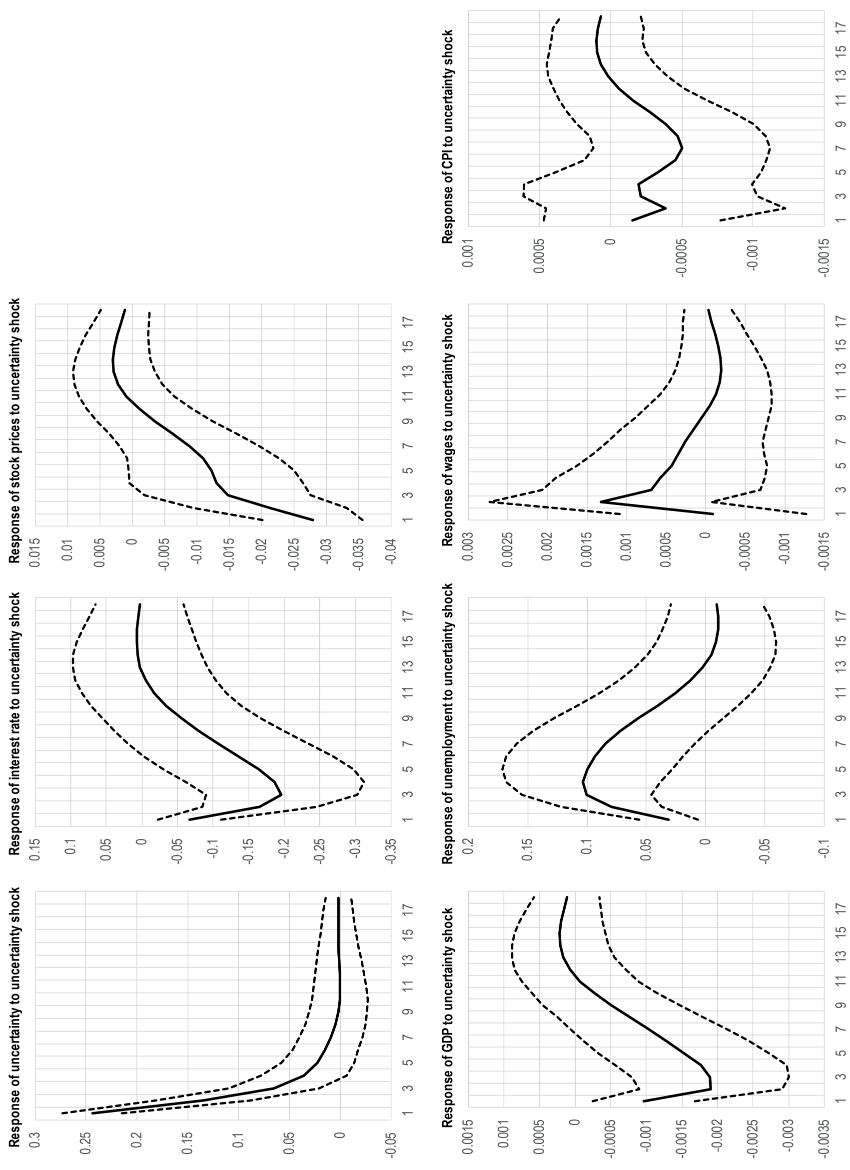


Figure A.3: Responses to uncertainty shock (EPUI – Monetary Policy)

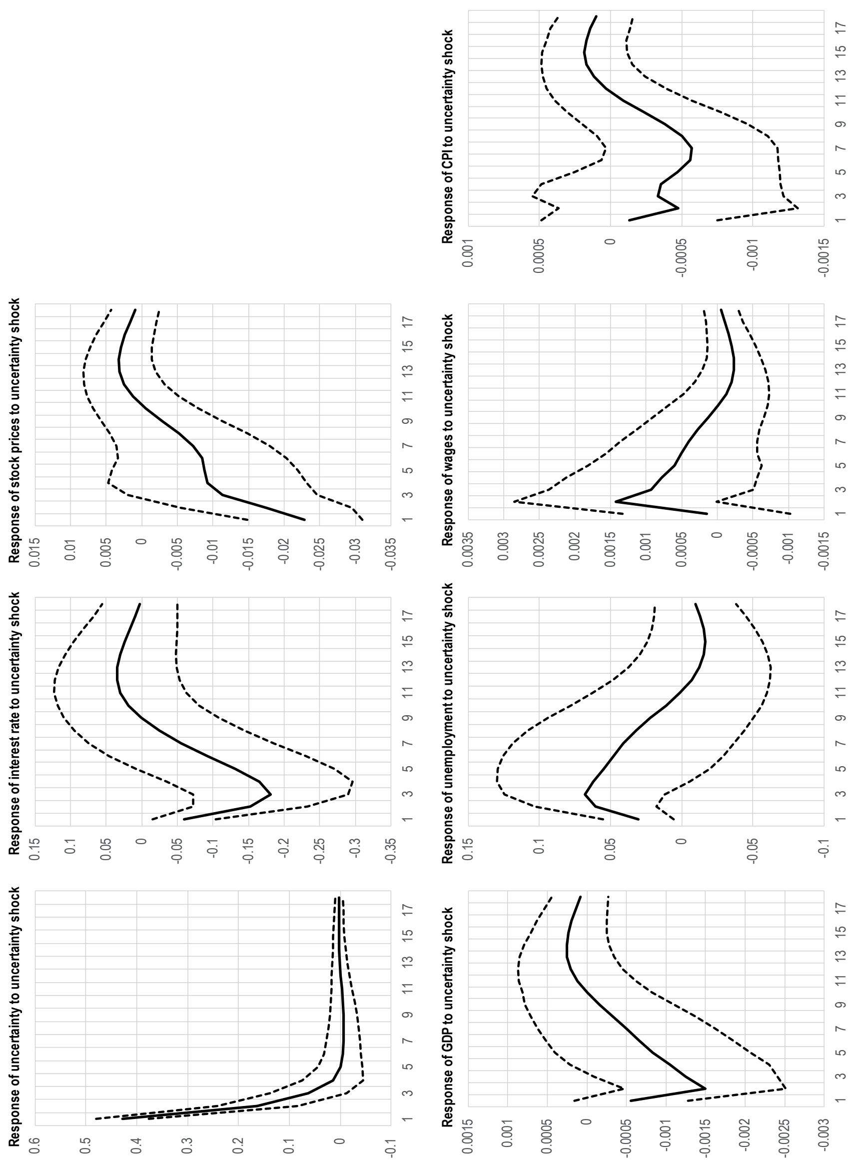


Figure A.4: Responses to uncertainty shock (JLN uncertainty index)

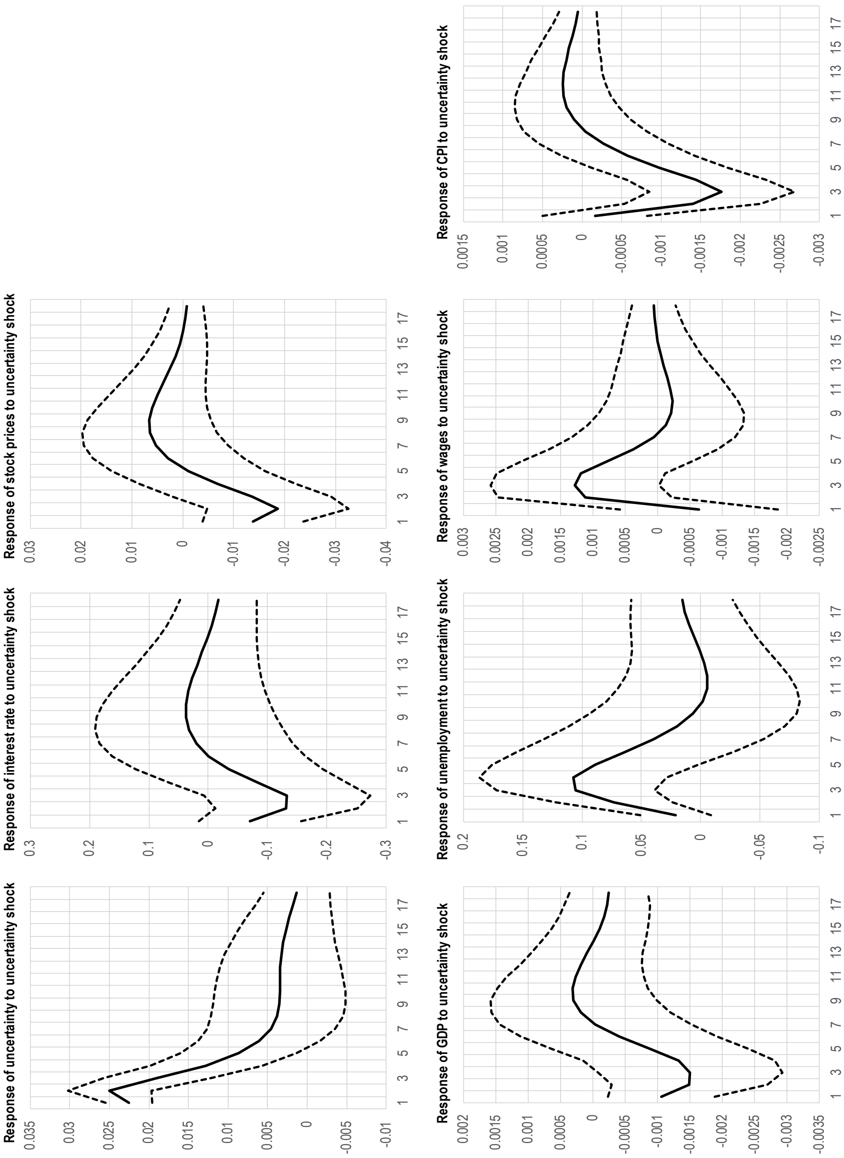


Figure A.5: Responses to uncertainty shock (forecast disagreement – GDP growth)

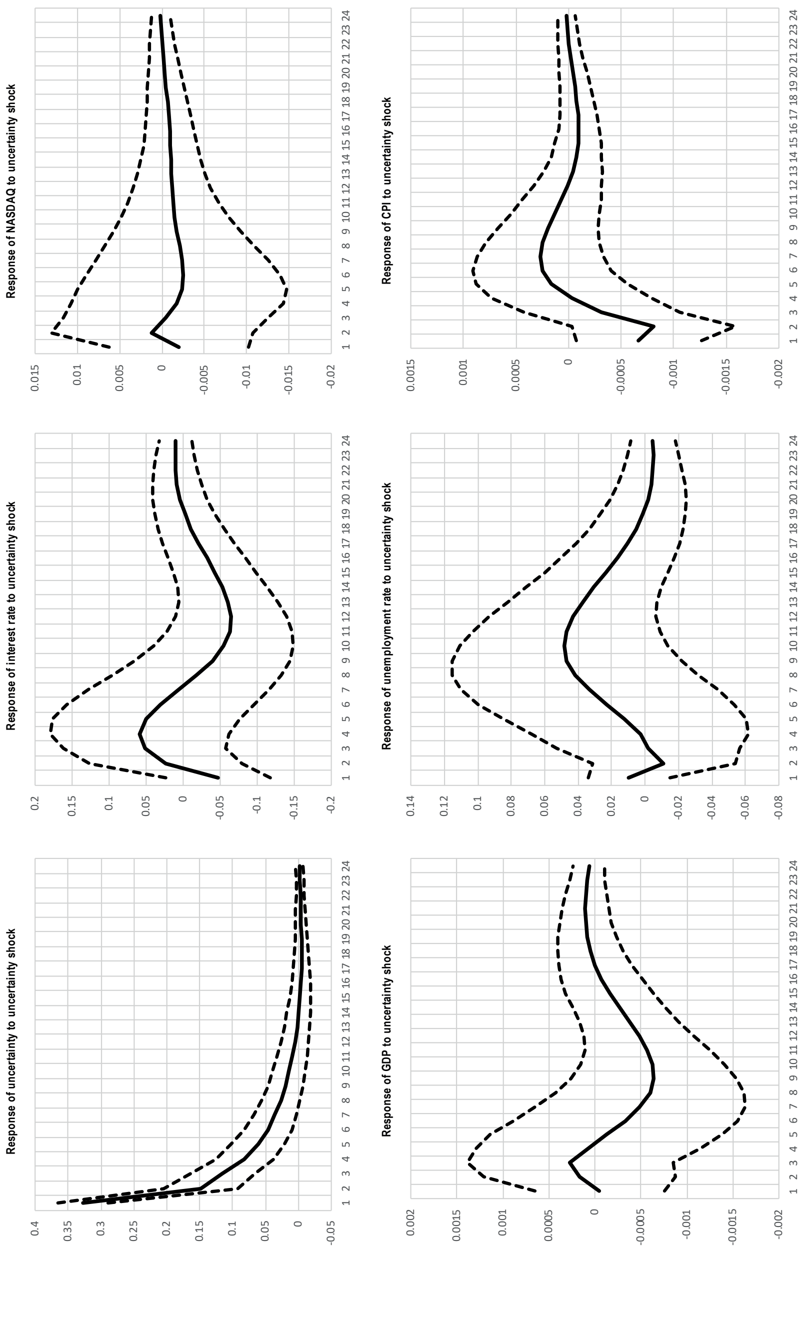
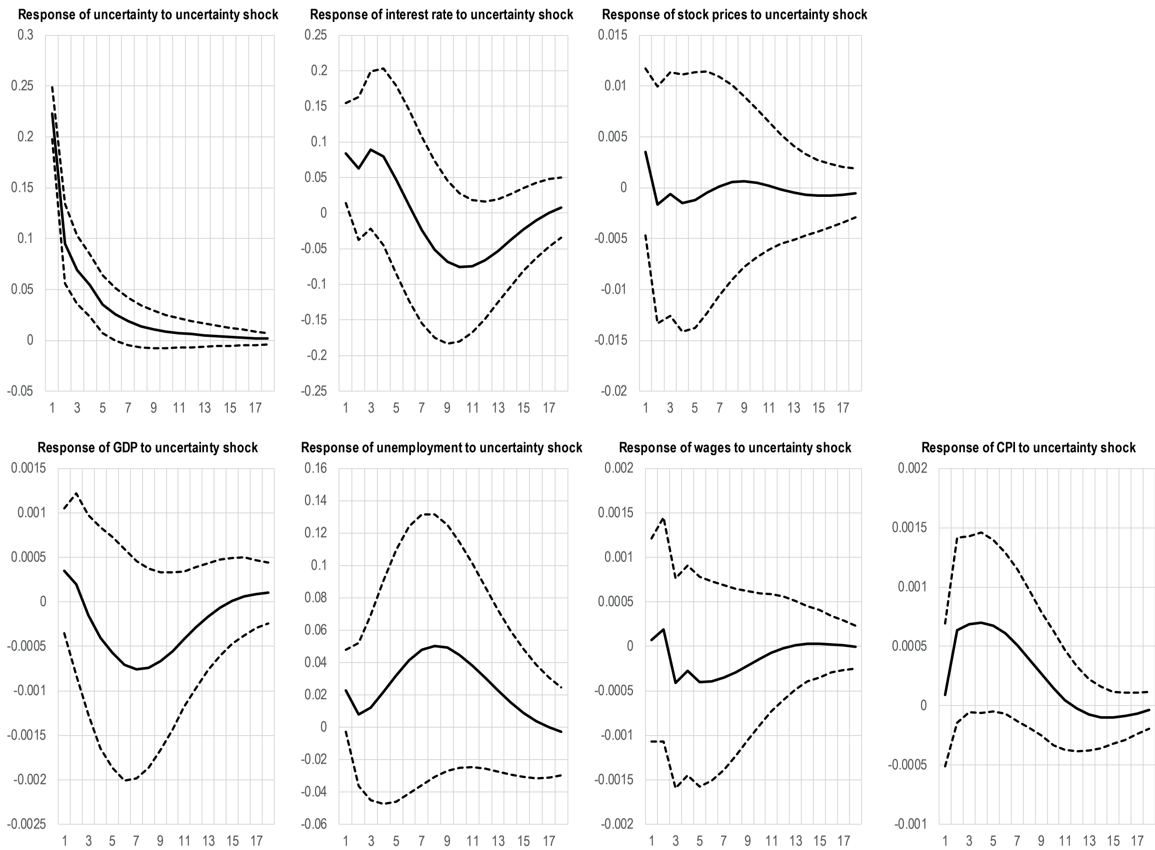


Figure A.6: Responses to uncertainty shock (forecast disagreement – inflation)



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