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Human Capital and Technological Progress as the Determinants of Economic Growth

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Foreword

The problem of economic growth is one of the fundamental fields of research in the theory of economics. Its essence is searching for answers to multiple questions about the causes and mechanisms of economic processes which lead to large cross-country differences in wealth between different countries and regions.

The classical convention in this type of research is to identify the determinants of growth and their impact on economic growth. This book concentrates on two basic factors of growth – human capital and technological progress, which supplement physical capital and labour. This choice is purposeful – human capital and technological progress are considered to be increasingly important from the point of view of economic growth and development. Focusing on these two factors of growth, with particular interest in technology diffusion, does not mean that one can not take into account such factors as: social, intellectual or cultural capital, institutions etc. On the contrary, it opens up new possibilities for research on economic growth and development.

This book is an interdisciplinary thesis on mathematical theory of economic growth and it is well-founded in the contemporary international economic literature. Its distinctive features are: innovativeness, originality, complexity and diversity of research tools, good choice of sources and statistical data and the use of original algorithms and software for empirical research and analysis of the considered models.

The Author has a profound knowledge on the contemporary economic growth theory and the rare abilities to creatively extend and apply the models and research methods originating from physics to economic phenomena. He formulates interesting and deep conclusions, but at the same time shows rational criticism towards his results.

The book should find due place in the world economic literature as a source of inspiration and an important contribution to the discussion on the choice of models, methods and techniques of interdisciplinary research on economic growth and development.

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Introduction

One of the most important questions of economics is the question about the nature and mechanism of economic growth. Very large differences in wealth across countries are an uncontested empirical fact. In the richest countries of the world the income per capita is more than a hundred times higher than in the poorest countries. What is the reason for this?

Such questions have been asked already by Adam Smith in "An Inquiry into the Nature and Causes of the Wealth of Nations" [72]. Smith claimed that the wealth of nations depends on the productive force of labour and the fraction of workers employed in productive sectors [54]. Both of these factors depend on the accumulation of capital. The output of the economy is divided between consumer goods and the accumulation of capital. The more output to the latter, the higher the growth rate of the economy. Moreover, to ensure an efficient capital accumulation, free market, private equity and unequal division of production are also vital.

The contemporary theory of economic growth was born in the 20th century, when Adam Smith's ideas were formalized, i.e. expressed in the language of mathematics. The pioneers of this area were Ramsey [65] in 1928 and Solow [73] and Swan [76] in 1956. In the models introduced by them, the key mechanism, i.e. the mechanism that ensured a balanced growth of quantities under consideration, was physical capital accumulation. Such mechanism can not, however, explain the phenomenon of economic growth perceived as the growth of output per worker. The emergence of economic growth is possible if we assume the existence of an additional production factor – knowledge embedded in technology (technological progress) or in human capital.

A commonly shared point of view is that the role of technological progress and human capital for long-term economic growth is of utmost importance. However, there is no common answer to the question how to take these factors into account when constructing actual models. In other words, the mechanism of human capital accumulation and technological progress is not known.

The simplest approach to technological progress is the assumption of exogenous technological progress. It can not, of course, elucidate the mechanism of technological progress, but it is capable of explaining the growth of per capita output in the Ramsey-Cass-Koopmans and Solow-Swan models.

In the 1980s, there have been first attempts at endogenizing technological progress and human capital accumulation. As the pioneers of this branch of economic growth theory, one considers Romer [67], [68] and Lucas [56]. The concepts introduced by them are now the basis for most of analyses of the role of technological progress and human capital for economic growth.

Since then, there have also been propositions of new models of these growth factors, especially ones trying to unify human capital and technological progress into a consistent framework. It seems reasonable now to name the crucial ideas of contemporary theories of technological progress and human capital.

The development of technology is usually modelled as the result of the so-called research and development (R&D) expenditures, i.e. it stems from deliberate human activity aiming at new technological solutions, which increase the efficiency of production. In addition, one often considers the transfer of technology (technology diffusion) among countries. It is quite common that some effective technological solutions (e.g. a computer) invented in one country are transferred or imitated in other countries. Therefore, low expenditures on R&D do not have to mean that the rate of technological progress has to be low. The pioneering work on technology diffusion was the one by Nelson and Phelps [61] in 1966.

On the other hand, the theory of human capital describes the accumulation of human capital as a result of two types of activities. First, there are deliberate investments in human capital, through schooling and training. Second, human capital is also accumulated on-the-job as the worker gains experience from doing his job.

The aim of this book is to consider various ways of theoretical modelling of technological progress and human capital in the context of the theory of economic growth. We will examine the consequences of different views on these concepts and we will analyze economic growth models which employ them. We will consider neoclassical models of economic growth, which originate from the Solow-Swan model, and endogenous growth models, in which economic growth results from some endogenous mechanisms, and also various modifications of these models. We will also propose an alternative way of modelling economic growth with the use of simulational models, employing the Monte Carlo simulation method.

An important part of this work is empirical research. The availability of statistical data made it possible to perform all analyses for the OECD countries in 1981-1999. The basic method of empirical verification that will be used throughout the book is the calibration of parameters of analyzed models. The essence of this method is to adjust a parameter or a group of parameters of a model in such a way that the resulting dynamics of some variable of the model (e.g. technology level, physical capital or output per worker) closely resembles empirical dynamics. The standard measure of adjustment in this work will be the mean absolute error, i.e. the average of absolute values of the difference between the model and empirical values. The calibration experiments will be conducted in two ways. Firstly, we can assume that some parameters of a model are equal in all countries. Thus, we can calibrate the value of this parameter in one country and then we can examine the dynamics of the model for other countries, having set the value of the parameter to its calibrated value. Secondly, if we presume that the value of some parameter is different among countries, we can calibrate this parameter for every country and draw conclusions from the calibrated values (e.g. the calibrated value of a diffusion parameter can tell us about the strength of technological diffusion processes in different countries).

The structure of the book is as follows.

In **chapter 1** we will review the current status of economic growth theory. We will show the mechanisms of economic growth in neoclassical models, starting from the Solow-Swan model and its modifications. We will also analyze the most important models of human capital and technological progress, which will be the basis for further considerations.

Chapter 2 will be devoted to the Manuelli-Seshadri model, which is one of the most advanced theoretical descriptions that deal with the human capital accumulation mechanism. The key element of this model is the qualitative aspect of human capital. Taking this aspect into account allows for a much more realistic modelling of this type of capital. The Manuelli-Seshadri model will also be analyzed empirically for the group of the OECD countries and we will reflect over the role of human capital in these countries.

In **chapter 3** we will show that much more satisfying results can be obtained if we add technological progress to the Manuelli-Seshadri model. We will also formulate and examine technological progress models with technology diffusion. All of the models will be verified empirically for the group of the OECD countries and we will conclude about the role of technology diffusion for economic growth in these countries.

In **chapters 4** and **5** we will introduce the simulational approach to technological progress and the whole economy. We will start with theoretical principles of the Monte Carlo method and move on to two simulational models of technological progress with technology diffusion and two economy models with the key role of technological progress and human capital. We will explain the essence of the simulational approach to the modelling of economic phenomena, which consists in the analysis of a group of heterogeneous economic agents with the help of computer tools. All of the models will also be verified empirically and we will draw conclusions about the use of this kind of models.

Finally, in the **conclusion** we will gather the most important conclusions from the analyses and we will discuss the possible directions for further work.

This book has been based on the Ph.D. thesis, defended at the Faculty of Economics of Poznań University of Economics in 2007[1]. I would like to thank my Advisor, Prof. Krzysztof Malaga for arousing my interest in economic growth theory and his constant and kind support and advice. I have also greatly profited from the weekly discussions at the Chair of Mathematical Economics seminars supervised by Professors Emil Panek and Zbigniew Czerwiński. I also thank Agnieszka and my Parents, to whom I dedicate this book.

1

Chapter 1

Review of economic growth, human capital, and technological progress models

In this chapter we will review the neoclassical models of economic growth, with particular interest in models employing the notions of human capital and technological progress. We will begin with the Solow-Swan model, which is the starting point for most of the contemporary considerations on economic growth.

1.1 The Solow-Swan model and similar models

1.1.1 The Solow-Swan model

This model has been proposed simultaneously by Solow [73] and Swan [76]. They considered the accumulation of a single aggregated good, whose production level is denoted by $Y(t)$. A fraction of production $sY(t)$ is saved (hence the parameter s is called the *savings rate*) and the remaining part ($(1-s)Y(t)$) is consumed. The accumulation of physical capital $K(t)$ is given by the following equation[1]:

$$\dot{K}(t) = sY(t) - \delta K(t), \tag{1.1}$$

i.e., the net increase in physical capital is equal to the difference between the saved part of production (investments) and the depreciation of physical capital (with a constant depreciation rate δ).

There are two factors of production in the model – *physical capital* $K(t)$ and *labour* $L(t)$. The production $Y(t)$ that can be obtained from a given combination of physical capital and labour is given by the *production function*:

$$F\left(K(t), L(t)\right) = Y(t). \tag{1.2}$$

We assume that the production function is homogeneous of degree 1 (there are constant returns to scale) and the population (the labour force) grows with a constant rate n:

$$L(t) = L_0 e^{nt}, \tag{1.3}$$

where L_0 is the initial population.

Let us introduce a new variable $k(t) \equiv K(t)/L(t)$, i.e. *physical capital per unit of labour*. We have:

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)} = \frac{sY(t)}{K(t)} - (n + \delta). \tag{1.4}$$

Constant returns to scale mean that $F(K, L) = LF(K/L, 1) \equiv Lf(k)$, hence, if we multiply equation (1.4) by $k(t)$, we obtain:

[1] All quantities with a dot denote the time derivative.

$$\dot{k}(t) = sf\,(k(t)) - (n+\delta)k(t). \tag{1.5}$$

Equation (1.5) is a differential equation in one variable $k(t)$.

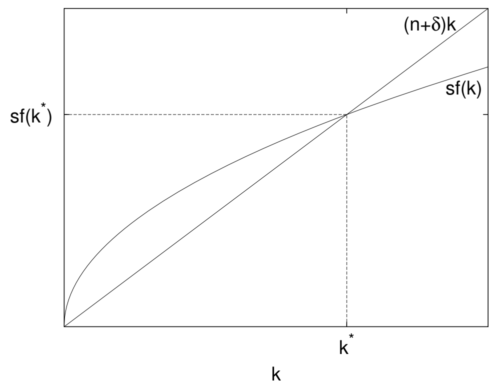
Let us now assume that the production function is differentiable, increasing, has decreasing marginal products to capital and labour (i.e. the second derivatives with respect to K and L are negative) and satisfies the Inada conditions[2]:

$$\lim_{K\to 0} \frac{\partial F}{\partial K} = \lim_{L\to 0} \frac{\partial F}{\partial L} = \infty, \tag{1.6}$$

$$\lim_{K\to \infty} \frac{\partial F}{\partial K} = \lim_{L\to \infty} \frac{\partial F}{\partial L} = 0. \tag{1.7}$$

The production function which satisfies all of the above assumptions is called the neoclassical production function[3].

Figure 1.1: **The steady state value of physical capital p.c. k^* in the Solow-Swan model**



In Fig. 1.1 we show the graphical illustration of eq. (1.5). The straight line expresses the effective depreciation of physical capital with a rate $n+\delta$ and the curve $sf(k)$ denotes the saved fraction of production for an exemplary neoclassical production function. The balance of these two effects determines the so-called *steady state* of the economy[4]. In this state, the net increase in physical capital per capita (p.c.) is zero ($\dot{k}=0$), i.e. the physical capital grows with the same rate as the labour force (n). The steady-state (or *stationary*) value of physical capital p.c. is denoted k^* on the plot. The steady-state increases in consumption p.c. $c(t) \equiv C(t)/L(t)$ and production p.c. $y(t) \equiv Y(t)/L(t)$ are also zero, thus the total consumption and production also grow with rate n. Later in this book we will use the term *steady-state* in a similar fashion, i.e. it will describe a situation when all the relevant variables grow with the same rate.

If $k \neq k^*$, physical capital p.c. will not be constant ($\dot{k} \neq 0$) – if $k < k^*$, investments will be larger than depreciation and physical capital p.c. will grow until it reaches the stationary value; if $k > k^*$, then depreciation will

[2]For the sake of simplicity, we sometimes leave out the arguments of functions, i.e. we denote $K(t) \equiv K$, $L(t) \equiv L$, etc.

[3]By neoclassical production function we also mean a three-factor function (usually with human capital as the third factor), which is continuous, differentiable, increasing, has decreasing returns to all factors of production, but constant returns to scale and satisfies the Inada conditions for all factors.

be larger than investments and physical capital p.c. will decrease towards k^*. This means that the steady-state in the Solow-Swan model is *stable*, i.e. deviations from this state will push the economy towards this state.

An important property of the Solow-Swan model is the lack of dependence of the growth rate of p.c. values (production, consumption, physical capital) on the technology of production (i.e. the form of the production function) and on the parameter values – growth rate of population, savings rate and depreciation rate. Thus, the sustainable growth of per capita variables is not possible (the growth rates of these variables are zero in the steady-state). In this way, *this version of the Solow-Swan cannot be used to model the long-term economic growth*. Changes in the parameter values (n, s, δ) can only lead to a change of the stationary value of physical capital p.c., production p.c. and consumption p.c.. If then the values of k, c and y are not equal to their steady-state values, the adjustment process towards these values will take place.

1.1.2 The Cobb-Douglas production function

The most widely used neoclassical production function is the *Cobb-Douglas production function* [24] of the following form:

$$F(A, K, L) = AK^{\alpha}L^{1-\alpha}, \tag{1.8}$$

where $A > 0$ denotes the level of technology and $\alpha \in (0, 1)$ is a constant.

The notion of the Cobb-Douglas production function is sometimes extended to all multiplicative production functions of the form:

$$F(A, K, L) = AK^{\alpha}L^{\beta}, \tag{1.9}$$

where $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ are constants[5]. If $\alpha + \beta < 1$, we have decreasing returns to scale, if $\alpha + \beta > 1$, we speak of increasing returns to scale, and the case $\alpha + \beta = 1$ is the classical Cobb-Douglas function with constant returns to scale.

In the book, the Cobb-Douglas and multiplicative (1.9) production functions are the basic functions used in the analyses.

1.1.3 The Solow-Swan model with technological progress

One of the easiest ways to take economic growth into account within the framework of the Solow-Swan model is to introduce the time dependence in the production function:

$$Y(t) = F(K(t), L(t), t), \tag{1.10}$$

which is equivalent to assuming *exogenous technological change*. The form of eq. (1.10) determines the character of technological progress. One usually distinguishes three cases (Barro, Sala-i-Martin [9]):

- Hicks-neutral technological change [39]:

$$F(K(t), L(t), t) = A(t)F(K(t), L(t)), \tag{1.11}$$

[4] Formally, the two effects are also balanced for $k = 0$ – leading, however, to a trivial steady state with zero production.

[5] Interpreted as the elasticities of production with respect to a given production factor.

- Harrod-neutral technological change [37] (also called labour-oriented):

$$F\left(K(t), L(t), t\right) = F\left(K(t), A(t)L(t)\right), \tag{1.12}$$

- Solow-neutral technological change [74] (also called capital-oriented):

$$F\left(K(t), L(t), t\right) = F\left(A(t)K(t), L(t)\right), \tag{1.13}$$

where the function $A(t)$ is called the technology index and can be interpreted as e.g. the accumulated knowledge, useful from the point of view of production processes.

In the original paper by Solow [73], the Hicks-neutral technological progress is considered. It is possible to show (e.g. Jones, Scrimgeour [47]) that in the case of the Cobb-Douglas production function (used by Solow), the forms (1.11), (1.12) and (1.13) of technological progress are equivalent. In the further course of this chapter we present the analysis of the Solow-Swan model with Harrod-neutral exogenous technological progress. Assuming this, we can proceed with any neoclassical production function – one can show that if technological progress is Harrod-neutral, then a steady-state exists in the Solow-Swan model (e.g. Jones, Scrimgeour [47]).

The physical capital dynamics equation takes the form:

$$\dot{K}(t) = sF\left(K(t), A(t)L(t)\right) - \delta K(t), \tag{1.14}$$

where the exogenous technology growth rate equals g, i.e.:

$$A(t) = A(0)e^{gt}, \tag{1.15}$$

and the growth rate of population equals n.

Let us now introduce new variables (variables *per effective unit of labour*):

$$\hat{k}(t) \equiv \frac{K(t)}{A(t)L(t)}, \tag{1.16}$$

$$\hat{c}(t) \equiv \frac{C(t)}{A(t)L(t)}, \tag{1.17}$$

$$\hat{y}(t) \equiv \frac{Y(t)}{A(t)L(t)}. \tag{1.18}$$

We then have:

$$\frac{\dot{\hat{k}}(t)}{\hat{k}(t)} = \frac{\dot{K}(t)}{K(t)} - g - n, \tag{1.19}$$

$$\hat{y}(t) = F\left(\hat{k}(t), 1\right) \equiv f\left(\hat{k}(t)\right). \tag{1.20}$$

Substituting now eq. (1.14) into eq. (1.19) and taking eq. (1.20) into account, we obtain an equation analogous to (1.5):

$$\dot{\hat{k}}(t) = sf\left(k(t)\right) - (g + n + \delta)\hat{k}(t), \tag{1.21}$$

which is a differential equation in a single variable $\hat{k}(t)$. The graphical illustration of this equation is also analogous – instead of physical capital p.c. $k(t)$ one has, however, physical capital per effective unit of labour in Fig. 1.1 and the effective depreciation rate is $g + n + \delta$, and not $n + \delta$. Thus, one obtains the value of physical capital per unit of effective labour unit $\hat{k}^*$ in the steady-state.

In the steady-state, the per unit of effective labour quantities are constant – $\hat{k}^*$, $\hat{c}^*$ and $\hat{y}^*$. This means that the p.c. variables (k, c and y) grow with the rate g. In this way, we have economic growth in the economy under consideration, where we understand this growth as the growth of production per capita. Economic growth in such model is fully exogenous and the growth rates of technology and production are equal to each other. This model does not tell us anything about the source of technological progress – therefore it contributes relatively little to the comprehension of the essence of the phenomenon of economic growth. Its value in the analysis consists in establishing the framework for the discussion on economic growth. It also enables empirical analyses, which are usually performed with the Cobb-Douglas production function of the form (1.8) and with the production p.c. dynamics equation obtained by substituting $\hat{k}^*$ in the steady-state:

$$\hat{k}^* = \left(\frac{s}{g+n+\delta}\right)^{1/(1-\alpha)} \tag{1.22}$$

into the production function (1.8) and taking the logarithm:

$$\ln y(t) = \ln A(0) + gt - \frac{\alpha}{1-\alpha}\ln(g+n+\delta) + \frac{\alpha}{1-\alpha}\ln s. \tag{1.23}$$

Most of the economic growth models that emerged after the Solow-Swan model refer directly or indirectly to this model, making it thus still the frame of reference in the discussion on economic growth.

1.1.4 The AK model

The simplest model of endogenous economic growth, i.e. growth whose determinants are included in the model, is the so-called *AK model*[6]. Here, we will analyze the simplest of a group of AK models, all of which have constant marginal products of physical capital. The production function is:

$$F\left(K(t), L(t)\right) = AK(t), \tag{1.24}$$

where $A > 0$ is a parameter, which can be interpreted as the level of technology.

Production per capita is:

$$y(t) = f\left(k(t)\right) = Ak(t), \tag{1.25}$$

and the physical capital p.c. dynamics equation takes the form:

$$\frac{\dot{k}(t)}{k(t)} = sA - \delta - n. \tag{1.26}$$

This means that if the growth rate of physical capital p.c. is constant and equal to $sA-\delta-n > 0$, then the physical capital p.c. and production p.c. will grow – we will therefore have economic growth even without technological progress. The growth rate of physical capital p.c., thus also of production p.c., does not depend on the level of physical capital – the economy always grows at the same rate, which depends only on the exogenous parameters s, A, δ and n.

[6] If we take the elasticity with respect to physical capital in the Cobb-Douglas production function equal to 1, then such production function describes production processes for which the production $Y(t)$ is a linear function of the physical capital $K(t)$. This means that there are constant returns to scale with respect to physical capital.

Solving the differential equation (1.26), we get the expression for the time dependence of physical capital and production p.c. If the initial value of physical capital is $k(0)$, the production p.c. at time t is given by:

$$y(t) = Ak(0)e^{(sA-\delta-n)t}. \tag{1.27}$$

The source of economic growth in the AK model is the absence of decreasing marginal productivity with respect to physical capital. This suggests that this capital should be interpreted broadly – not as pure physical capital, but rather as the sum of physical and human capital. The assumption of non-decreasing marginal productivity with respect to physical capital does not seem to be plausible. Similarly as in the Solow-Swan model with technological progress, the source of sustainable economic growth is the *growing knowledge resource*, embodied in technological progress (in the Solow-Swan model) or in human capital (which emerges indirectly in the AK model).

1.1.5 The Ramsey-Cass-Koopmans model

In this subsection we will deal with the Ramsey-Cass-Koopmans model [65], [20], [49] in which the savings rate s, the key parameter in the Solow-Swan model, is determined endogenously as the result of rational behaviour of households who optimize their activity.

We consider a representative household, whose members want to maximize the total utility of the household – their own utility and the utility of future generations, who will be the members of the household in the future (in infinite time horizon). The total utility U of the household at the initial moment is given by the expression:

$$U(0) = \int_0^\infty u\left(c(t)\right) e^{-(\rho-n)t} dt, \tag{1.28}$$

where $u\left(c(t)\right)$ denotes the instantaneous utility function for consumption p.c. $c(t)$ at time t, $\rho > 0$ is the discount rate of the household, $n > 0$ the growth rate of the household's population (we assume that $L(0) = 1$, i.e. $L(t) = e^{nt}$). The effective discount rate is $\rho - n$, because the household's utility comes also from the consumption of a growing number of people who form it. We assume that $\rho > n$, i.e. the effective discount rate is $\rho - n > 0$. About the instaneous utility function we assume that it is increasing, twice differentiable, strictly concave ($u'(c) > 0, u''(c) < 0$) and satisfies the Inada-type conditions:

$$\lim_{c\to 0} u'(c) = \infty, \tag{1.29}$$

$$\lim_{c\to\infty} u'(c) = 0. \tag{1.30}$$

The production in the economy is given by an aggregative, neoclassical production function without technological progress:

$$Y(t) = F\left(K(t), L(t)\right). \tag{1.31}$$

Similarly to the Solow-Swan model, we define per capita variables ($k \equiv K/L$, $y \equiv Y/L = F(K,L)/L = F(K/L, 1) \equiv f(k)$).

We assume that the capital and labour markets are perfectly competitive, i.e. the return rate on capital is equal to the difference of marginal product of capital and its depreciation rate:

$$r(t) = f'\left(k(t)\right) - \delta, \tag{1.32}$$

where $r(t)$ denotes the risk-free interest rate, δ the depreciation rate of capital, and the wage per worker $w(t)$ equals the marginal product of labour:

$$w(t) = \frac{\partial F\left(K(t), L(t)\right)}{\partial L(t)} = f\left(k(t)\right) - k(t)f'\left(k(t)\right), \tag{1.33}$$

where the last equality results from the fact that constant returns to scale imply:

$$F\left(K(t), L(t)\right) = \frac{\partial F\left(K(t), L(t)\right)}{\partial K(t)}K(t) + \frac{\partial F\left(K(t), L(t)\right)}{\partial L(t)}L(t). \tag{1.34}$$

This is the so-called Euler's identity.

The representative household has assets $A(t) = a(t)L(t)$, where $a(t)$ are assets p.c. The assets dynamics equation is:

$$\dot{A}(t) = r(t)A(t) + w(t)L(t) - c(t)L(t), \tag{1.35}$$

where the subsequent terms on the right-hand side represent income from the ownership of assets, income from work and consumption expenses, respectively. Using p.c. variables, we have:

$$\dot{a}(t) = \left(r(t) - n\right)a(t) + w(t) - c(t). \tag{1.36}$$

The effective return rate on assets equals $r(t)-n$, since at later time moments the assets $A(t)$ are divided by a larger number of people $L(t)$. We assume that assets p.c. of a household are equivalent to its physical capital p.c. that can be used in production. Thus:

$$a(t) = k(t). \tag{1.37}$$

We also assume:

$$\lim_{t\to\infty} a(t)e^{-\int_0^t (r(s)-n)ds} \geq 0, \tag{1.38}$$

which means that in the infinite time horizon the household can not have negative assets[7] (allowing for negative assets would mean that the current, arbitrarily large consumption could be financed from future, arbitrarily large debt, which would grow faster than $r(t)$).

The utility maximization problem (1.28), together with the assets dynamics equation (1.36) and the boundary condition (1.38) can be solved with the optimal control techniques.

Let us assume the CIES[8] consumption function:

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta}, \tag{1.39}$$

where θ is the inverse of the elasticity of substitution between consumption at two arbitrary time moments[9].

The first-order conditions imply:

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}\left(r(t) - \rho\right). \tag{1.40}$$

[7]The transversality condition, which determines the solution at the final moment, implies that (1.38) must be satisfied with equality.

[8]CIES – Constant Intertemporal Elasticity of Substitution. The CIES function has a constant intertemporal elasticity of substitution $1/\theta$.

[9]$\theta = -u''(c)c/u'(c)$.

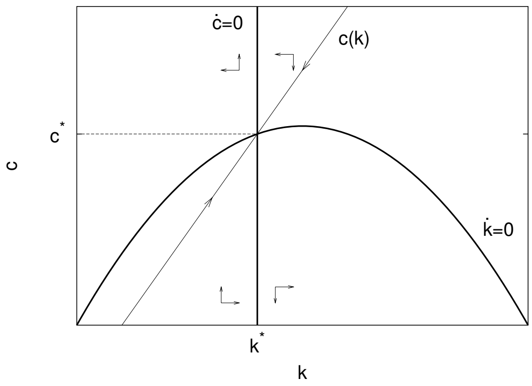
Using eq. (1.32), we obtain:

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}\left(f'\left(k(t)\right) - \delta - \rho\right). \tag{1.41}$$

Equations (1.36), (1.37) and (1.33) imply the following differential equation for $k(t)$:

$$\dot{k}(t) = f\left(k(t)\right) - (n+\delta)k(t) - c(t). \tag{1.42}$$

Figure 1.2: **The phase diagram of the Ramsey-Cass-Koopmans model**



In the steady-state, we have $\dot{c}(t) = 0$. The value of physical capital p.c. in the steady-state is k^* and satisfies:

$$f'\left(k^*\right) = \rho + \delta. \tag{1.43}$$

The consumption in the steady-state, according to eq. (1.42), is:

$$c^* = f\left(k^*\right) - (n+\delta)k^*. \tag{1.44}$$

Fig. 1.2 shows the phase diagram of the Ramsey-Cass-Koopmans model and illustrates the properties of this model.

1. The steady-state consumption p.c. c^* is not the maximal level of consumption that could be reached, which is due to the non-zero value of the discount rate – the current consumption is valued more than one's own future consumption and the consumption of future generations.

2. There exists a line $c(k)$ (the so-called *stability branch*) such that if at a given time the point (k, c) lies on this curve, then the economy moves towards the steady-state (k^*, c^*).

3. If at a given time, the point (k, c) does not lie on the stability branch, then the economy will move towards either $c = 0$, or $k = 0$. Such situations violate, however, the first-order conditions ($k = 0 \Rightarrow f(k) = 0 \Rightarrow c = 0$) or the transversality condition (zero consumption for $k \neq 0$ means that the household's utility can not be the maximal one).

Introducing exogenous technological change does not change the conclusions substantially. The analyzed quantities are again the per effective unit of labour variables and one obtains a steady-state in which the p.c. quantities grow with the growth rate of technology.

The Solow-Swan model can be regarded as the special case of the Ramsey-Cass-Koopmans model with a constant savings rate s. In the Ramsey-Cass-Koopmans model, the savings rate $s = 1 - c/f(k)$ results from households' maximization problems. In the steady-state, however, the savings rate is constant and the Ramsey-Cass-Koopmans and Solow-Swan models are then equivalent from the practical point of view.

1.2 Models with human capital

The models that we analyzed in the previous section do not provide us with a satisfying answer to the question about the mechanism of economic growth. They form, however, the basis for further considerations on economic growth. Nowadays, the key role in economic growth is assigned to the *knowledge accumulation processes* and one introduces to the analysis the notions of *human capital* and *endogenous technological change*.

In this section we will analyze models in which human capital is the key factor of growth. In particular, we will show the evolution of these models towards the more advanced theoretical formulations, like the Manuelli-Seshadri model, which will be the main topic of the next chapter.

One of the first economists who noticed that a human being can be treated as a form of capital was Adam Smith, who considered the capabilities of the citizens of a country to be a part of the capital of this country [71]. In this way, he could explain why some nations are richer than some other. The key idea that underlies the modern human capital-oriented economic growth models is similar.

First attempts at a formalization of this idea emerged only in the 1950s and 1960s in the works of e.g. Mincer [60], Schultz [71], Arrow [5], Becker [13] and Weisbrod [79]. In the above-mentioned papers the authors introduced representative individuals who invested in their development (through schooling or on-the-job training (*learning-by-doing*)) and thus improving their value on the labour market. Human capital was therefore perceived as the set of abilities of a worker that could be useful in the productive processes. The workers' gain from their improved efficiency at work is a higher wage rate. This gain can be viewed upon as the compensation for a longer schooling period, which entails that a person enters the labour market at a later stage of their lives and in this way loses a part of their potential income. The longer learning process also requires some additional expenses. The relation between these effects is the common property of nearly all the human capital models – human capital investments are profitable only up to the point where its positive (higher wage rate and thus higher potential consumption) and negative influence (postponed entrance to the labour market and additional learning expenses) balance.

1.2.1 The Ben-Porath model

The Ben-Porath model [17] of 1967 is the first model in which there is explicit human capital accumulation as the result of maximization of the sum of discounted earnings of a representative individual. The individual can devote their time either to work, or to human capital investments. Let us denote the individual's human capital at time t as $H(t)$. We assume that the representative individual has some initial human capital endowment $H(0)$, which is depreciated with rate δ. Human capital does not contribute to the individual's utility (i.e. is not an argument of the utility function), but it influences the individual through a possible increase of wages, which can in turn increase the consumption:

$$Y(t) = \alpha_0 H(t), \tag{1.45}$$

where α_0 is a parameter and $Y(t)$ denotes the individual's earnings at time t. Disposable earnings $E(t)$ are:

$$E(t) = Y(t) - I(t), \tag{1.46}$$

where $I(t)$ denotes human capital investments at time t. The human capital production function takes the form:

$$Q(t) = \beta_0 \left(s(t)H(t)\right)^{\beta_1} D(t)^{\beta_2}, \tag{1.47}$$

where $Q(t)$ denotes human capital accumulated at time t, β_i are positive parameters ($\beta_1 + \beta_2 < 1$), $D(t)$ expresses the quantity of goods used in the process of accumulation and $s(t)H(t)$ the human capital resource used in its further production ($0 \leq s(t) \leq 1$). Human capital dynamics equation is:

$$\dot{H}(t) = Q(t) - \delta H(t). \tag{1.48}$$

The cost of human capital investments consists of lost earnings and the cost of goods needed in the process of accumulation (with price P_d):

$$I(t) = \alpha_0 s(t)H(t) + P_d D(t). \tag{1.49}$$

One can easily show that the minimal investment to produce human capital $Q(t)$ equals:

$$I(t) = \frac{\beta_1 + \beta_2}{\beta_1}\alpha_0 \left(\frac{\beta_1 P_d}{\beta_2 \alpha_0}\right)^{\frac{\beta_2}{\beta_1+\beta_2}} \left(\frac{Q(t)}{\beta_0}\right)^{\frac{1}{\beta_1+\beta_2}}. \tag{1.50}$$

The aim of the individual is to maximize the current value $W(t)$ of disposable earnings:

$$W(t) = \int_t^T e^{-rx}\left(\alpha_0 H(x) - I(x)\right) dx, \tag{1.51}$$

where T is the retirement age and r the interest rate.

The maximization problem of $W(t)$ can be solved with optimal control techniques with the fraction of time devoted to human capital accumulation $s(t)$ as the control variable. The condition $0 \leq s(t) \leq 1$ specifies three phases in the lifetime of an individual. In the first phase, the upper limit for $s(t)$ is important, i.e. $s(t) = 1$ and the whole human capital is allocated to its further production. The individual does not work at this time – hence this is the period of formal education. In the second phase $0 < s(t) < 1$, i.e. the individual still learns, but they also use what they have learned so far in their work. In the third phase $s(t) = 0$, which means that human capital accumulation is no longer profitable and the individual can use all of their human capital in their work. Such three phases will also be present in more complex models, analyzed in further parts of this book.

For the first phase, it is not possible to give analytical expressions for the quantity of produced human capital $Q(t)$. The age at which the individual starts to work $(s(t) < 1)$ is denoted t^* and depends on the initial conditions. A larger initial human capital resource $H(0)$ implies earlier entrance into the phase when human capital is both produced and used on-the-job.

In the second phase, one defines the marginal cost of human capital production as the derivative of the investment cost (1.50) with respect to $Q(t)$ and the marginal profit as the discounted value of the increase in earnings due to the human capital investment. The optimal value of human capital production is found by equating the marginal cost and profit related to an additional unit of human capital:

$$Q(t) = \beta_0 \left(\frac{\beta_0\beta_1}{r+\delta}\right)^{\frac{\beta_1+\beta_2}{1-\beta_1-\beta_2}} \left(\frac{\beta_2\alpha_0}{\beta_1 P_d}\right)^{\frac{\beta_2}{1-\beta_1-\beta_2}} \left(1 - e^{-(r+\delta)(T-t)}\right)^{\frac{\beta_1+\beta_2}{1-\beta_1-\beta_2}}. \tag{1.52}$$

The form of the solution implies that human capital production takes place for every $t < T$. Hence, phase three occurs only at the point $t = T$.

By calculating the derivative of $Q(t)$ with respect to time, one can show that the rate of human capital production slows down when the individual ages. Depending on the initial human capital and its depreciation rate, the net increase in human capital becomes negative at a certain age (in particular, it is possible also at the initial time for some values of the parameters).

There are no direct links between the Ben-Porath model and the phenomenon of economic growth. Its meaning for the economic growth theory consists in showing how human capital investments can be described formally. The Ben-Porath model is also one of the first theoretical attempts to show how an individual's human capital can result from their conscious and rational decisions (disposable earnings maximization). Similar ideas were later used in many other models of economic growth, including the Manuelli-Seshadri model, which is the basis for considerations in the following chapter.

1.2.2 The Lucas model

Since the late 1960s (the Ben-Porath model) until the late 1980s few original models of human capital emerged.

A new strand in human capital research was started by Robert Lucas' work [56] of 1988. The author was reflecting upon a construction of a neoclassical theory of economic growth and international trade, a theory which would be in accordance with empirical data on economic growth. Lucas considered three models, which revealed different aspects of the issue. The most important one is considered to be the model of human capital accumulation through schooling. This model is in many ways similar to the Ben-Porath model, but here human capital is an important factor of economic growth. Lucas referred to the models of Solow [73], Schultz [71], Becker [13] and Uzawa [78].

Human capital was understood as the overall abilities of an individual which influenced their productivity.

Lucas considered an economy, in which the number of workers grows exogenously with a constant rate $\lambda > 0$:

$$N(t) = N(0)e^{\lambda t}. \tag{1.53}$$

Every worker is endowed with some resource of human capital $h(t) \in [0, \infty)$. A worker with human capital $h(t)$ allocates a fraction $u\left(h(t), t\right)$ of their time for current production and the remaining fraction $1 - u\left(h(t), t\right)$ for human capital accumulation. The effective labour force $N^e(t)$ depends on the quantity of the workers' human capital and on how it is allocated:

$$N^e(t) = \int_0^\infty u\left(h(t), t\right) N(h, t) h(t) dh(t), \tag{1.54}$$

where $N(h, t)$ denotes the number of workers with human capital h at time t.

The average human capital per worker $h_a(t)$ equals:

$$h_a(t) = \frac{1}{N(t)} \int_0^\infty h(t) N(h, t) dh(t). \tag{1.55}$$

To simplify the analysis, one assumes that all workers are identical (hence, a representative worker emerges in a similar way as in other neoclassical models), have the same human capital endowments:

$$h_a(t) = h(t) \tag{1.56}$$

and allocate a fraction $u(t)$ to current production. The amount of effective labour is thus given by:

$$N^e(t) = u(t) h(t) N(t). \tag{1.57}$$

The balance condition of the economy at time t takes the form:

$$N(t)c(t) + \dot{K}(t) = AK(t)^\beta \left(u(t)h(t)N(t)\right)^{1-\beta} h_a(t)^\gamma, \tag{1.58}$$

where $c(t)$ denotes consumption, $K(t)$ – physical capital, A – a constant level of technology, β and γ are parameters. The national income in this economy is divided between consumption and investments (the left-hand side of the above equation) and production is based on a given technology, described by the right-hand side of equation (1.58). The factor $h_a(t)^\gamma$ expresses the human capital externality. The average level of human capital influences the productivity, but no individual decision of an individual can influence this level (we assume that $N(t)$ is large), so it is not taken into consideration in allocation decisions.

The human capital dynamics equation takes the form:

$$\dot{h}(t) = h(t)^\zeta G\left(1 - u(t)\right), \tag{1.59}$$

where ζ is a parameter and $G\left(1 - u(t)\right)$ an increasing function of time allocated to human capital accumulation $1 - u(t)$, where $G(0) = 0$.

To simplify again, one assumes[10] that $\zeta = 1$ and the function $G(1 - u(t))$ is linear. We thus obtain:

$$\dot{h}(t) = h(t)\delta\left(1 - u(t)\right), \tag{1.60}$$

where δ is a parameter which expresses the maximal production rate of human capital. The human capital dynamics equation in such form implies constant returns to human capital, i.e. a given relative increase in $h(t)$ requires an equal effort, regardless of the current level of $h(t)$.

[10] Lucas accepted such an assumption, because it is in agreement with empirical data, which was shown by Rosen in 1976 [69].

In the Lucas model the consumers' preferences are described by a CIES-type utility function (1.39), hence the aggregated utility of consumption equals (by analogy with (1.28)):

$$U = \int_0^\infty e^{-\rho t}\frac{c(t)^{1-\theta}-1}{1-\theta}N(t)dt, \tag{1.61}$$

where ρ is the discount rate. The aim of the representative individual is to maximize their total utility (1.61), subject to (1.56), (1.58) and (1.60).

The set of functions $K(t)$, $h(t)$, $h_a(t)$, $c(t)$ and $u(t)$ such that the utility (1.61) is maximal, subject to (1.56), (1.58) and (1.60), describes the *optimal growth path* in the Lucas model. Using optimal control methods, one can show that the optimal growth rate of human capital ν^* equals:

$$\nu^* = \frac{1}{\theta}\left(\delta - \frac{1-\beta}{1-\beta+\gamma}(\rho-\lambda)\right), \tag{1.62}$$

under the condition $\nu^* \leq \delta$, which implies the following inequality for the model parameters:

$$\theta \geq 1 - \frac{1-\beta}{1-\beta+\gamma}\frac{\rho-\lambda}{\delta}. \tag{1.63}$$

In the Lucas model on can also define an *equilibrium path*. This is a set of functions $K(t)$, $h(t)$, $c(t)$ and $u(t)$ such that the utility (1.61) is maximal, subject to (1.58) and (1.60), and under the assumption that the path $h_a(t)$ is given exogenously and the representative individual expects that the average level of human capital will equal $h_a(t)$ and behave in such a way that the condition $h(t) = h_a(t)$ holds. Again, using optimal control techniques, one can show that the equilibrium growth rate of human capital ν equals:

$$\nu = \frac{1}{\theta(1-\beta+\gamma)-\gamma}(1-\beta)(\delta-(\rho-\lambda)), \tag{1.64}$$

under the condition $\nu \leq \delta$, which implies again (1.63).

If the condition (1.63) holds with equality, then:

$$\nu = \nu^* = \delta, \tag{1.65}$$

which means that the optimal and equilibrium growth rates of human capital are equal and maximal. Otherwise, $\nu < \nu^*$ always holds. The presence of human capital externality implies that the representative individual, who does not take this effect into consideration, invests less in human capital than would be optimal from the point of view of the whole economy.

One can also show that the growth rates of consumption and physical capital are equal to the same number κ:

$$\kappa = \left(\frac{1-\beta+\gamma}{1-\beta}\right)\nu. \tag{1.66}$$

If there is no human capital externality (i.e. if $\gamma = 0$), the growth rates of physical capital, human capital and consumption p.c. are equal. If the externality is non-zero ($\gamma > 0$), then physical capital grows faster than human capital.

The meaning of the Lucas model consists in showing that economic growth can be generated by investments in two types of capital. The accumulation mechanism is, however, very similar for these types of capital – this is why Lucas spoke of the *mechanics* of economic growth – he was modelling this phenomenon with the help of differential equations similar to the

Solow-Swan model equations. Many authors introduced similar models later on, including: Rebelo [66], Laitner [52], Caballe and Santos [19]). Such models make it possible to achieve a better agreement between the model and empirical data, since new variables and parameters are introduced and can be associated to some empirical facts about e.g. schooling or, more generally, the economic role of knowledge.

1.2.3 The Mankiw-Romer-Weil model

One of the most important and most widely used human capital models is the model of Mankiw, Romer and Weil (MRW) [58] of 1992. The MRW model is a modified version of the Solow-Swan model. The key novelty is the introduction of human capital as an additional production factor. The production function is of the form:

$$Y(t) = K(t)^{\alpha} H(t)^{\beta} \left(A(t)L(t)\right)^{1-\alpha-\beta}, \tag{1.67}$$

in which standard notation is used for physical capital ($K(t)$), human capital ($H(t)$), the level of technology ($A(t)$) and the number of workers ($L(t)$). We assume that $\alpha + \beta < 1$, i.e. there are decreasing returns with respect to the total capital. Introducing the per effective unit of labour units (1.16)-(1.18) and human capital per effective unit of labour:

$$\hat{h}(t) \equiv \frac{H(t)}{A(t)L(t)}, \tag{1.68}$$

the dynamics of the economy is described by the following set of differential equations:

$$\dot{\hat{k}}(t) = s_k \hat{y}(t) - (g + n + \delta)\hat{k}(t), \tag{1.69}$$

$$\dot{\hat{h}}(t) = s_h \hat{y}(t) - (g + n + \delta)\hat{h}(t), \tag{1.70}$$

where the growth rates equal: g – for the level of technology, n – for population, the common depreciation rate of physical and human capital equals δ, and s_k, s_h denote the investment rates in physical and human capital, respectively.

Similarly as in the Solow-Swan model, the accumulation equations of physical (1.69) and human capital (1.70) imply that the economy tends towards a steady state, in which the level of these two types of capital are given by:

$$k^* = \left(\frac{s_k^{1-\beta} s_h^{\beta}}{g + n + \delta}\right)^{1/(1-\alpha-\beta)}, \tag{1.71}$$

$$h^* = \left(\frac{s_k^{\alpha} s_h^{1-\alpha}}{g + n + \delta}\right)^{1/(1-\alpha-\beta)}. \tag{1.72}$$

The dynamics of the MRW model economy is hence analogous to the dynamics in the Solow-Swan model, except for the fact that we have two types of capital here. The way of modelling of these two types of capital is, however, essentially the same. The introduction of human capital leads to a much better agreement between the model and empirical data. Thus, the MRW model is the basis for much empirical research about the role of human capital in economic growth and about economic growth itself. To this aim, one uses the following equation for production p.c. – $y(t)$ – analogous to equation (1.23) in the standard Solow-Swan model – obtained by substituting (1.71) and (1.72) into the production function (1.67) and taking the logarithm:

$$\ln y(t) = \ln A(0) + gt - \frac{\alpha+\beta}{1-\alpha-\beta}\ln(g+n+\delta) + \frac{\alpha}{1-\alpha-\beta}\ln s_k + \frac{\beta}{1-\alpha-\beta}\ln s_h. \quad (1.73)$$

An important difference with respect to the analogous equation in the Solow-Swan model is the emergence of an empirical measure for the investment rate in human capital s_h. To estimate s_h, MRW used UNESCO data about the fraction of population aged 12 to 17 who attended high school, multiplied by the fraction of population aged 15 to 19. This variable was named SCHOOL. Together with other variables (all of which were statistically significant) – the ratio of investments to GDP and the sum $g+n+\delta$, the SCHOOL variable accounted for ca. 80% of GDP p.c. variability for the sample of 98 countries. MRW obtained also estimates for the parameter values: $\alpha \approx 0.3$ and $\beta \approx 0.3$. These estimates turned out to be compatible with the typical estimates of the elasticity of production with respect to total capital (around 2/3).

In comparison with the standard Solow-Swan model, the MRW model allows to obtain much better agreement with empirical data, especially if one uses more complex measures for human capital investments than the SCHOOL variable of MRW. The meaning of this model for the comprehension of the essence of economic growth and the role of human capital is therefore limited.

1.2.4 Other models

Most of the human capital models are in many respects similar to the ones described above. Here a brief overview of some alternative concepts is given.

In 1966 Nelson and Phelps [61] considered the relationship between technological progress and human capital. Human capital resources in this model determine the diffusion rate of technology between the technological leader and the country under analysis. The meaning of this sort of modelling is substantial and hence we will analyze it in detail in the next section.

The key role in the model of Becker, Murphy and Tamura [15] of 1990 is attributed to demographic variables, which influence human capital, which is understood as the knowledge accumulated in people. The authors of the model assumed that a higher level of human capital accelerates its further accumulation. However, investments in human capital are discouraged by high fertility. Population growth is endogenous and has its effect on the growth rate of economy. Countries with small initial human capital endowment and high fertility are unable to reach high growth rates – this can help to explain the observed cross-country differences in welfare. On the other hand, countries with high initial human capital grow faster, since it is then profitable to invest further in human capital and not in family enlargement. Other papers in which the relationship between human capital and fertility is considered are, among others, Becker and Barro [14] and Rosenzweig [70].

In 1990 Azariadis and Drazen [8] considered the role of threshold externalities in economic growth. They assumed that this kind of effects can emerge in human capital accumulation and lead to the existence of many locally stable steady-states. A switch between these steady-states is possible when some variable, e.g. literacy, reaches some threshold value. The authors have also shown the results of empirical analysis that supported the conclusion that high investments in human capital are the prerequisite for economic growth.

A similar way of modelling of human capital was presented by Stokey [75] in 1991. The author emphasized the role of relationship between human capital, technological progress and international trade.

In subsection 1.3.3 we will describe the Jones model [43], which is one of the attempts to merge the analysis of human capital and technological progress in a coherent theoretical formulation. It seems that such class of models is the most appropriate way to understand economic growth.

A basically different approach to human capital is the one of Hendricks [38] of 2002. He assumed that empirical data about immigrant earnings at the same labour market (USA) can help to estimate their levels of human capital. One usually supposes that all workers of the same age and the same level of education have equal human capital levels, independently of their country of origin. Such approach does not, however, take into account the existence of so-called immeasurable abilities. Hendricks assumed that if two workers have equal measurable abilities (education and experience) and their wages differ, then this difference in earnings results from immeasurable abilities. Thus, the notion of the quality of human capital emerged. Hendricks introduced this variable, but was unable to explain the cross-country differences in GDP p.c. with only differences in physical and human capital.

Apart from the above-mentioned papers, there are many papers with purely empirical analyses. The most widely investigated relationship is between the level/quality of schooling and economic growth (e.g. Bils and Klenow [18], Barro [11], Temple [77]). The theoretical basis for such research is, however, too simplified to tell much about the role of human capital in economic growth. Therefore, the models described in previous subsections, together with models that combine human capital and technological progress, seem to have a much larger potential.

1.3 Models of technological progress

As we mentioned in the previous section, an increasingly important role is attributed to the accumulation of knowledge, to which the notions of human capital and technological progress are related. In this subsection, we will consider models which view technological progress as the main growth factor. However, human capital is also an important factor in many of them. One often assumes that only the combination of technological progress and human capital effects can account for economic growth.

As in the case of human capital models, the first attempts at the formalization of technological progress emerged a few years after classic works of Solow [73] and Swan [76]. The most important of these are the papers of Uzawa [78], Phelps [64], Nelson and Phelps [61] and Nordhaus [62].

1.3.1 The Nelson-Phelps and Benhabib-Spiegel models

Let us assume that the level of technology at time t in the i-th country is given by the variable $A_i(t)$. Thus, there exists a country with the highest level of technology. Let us denote the technology level of the technological leader as $T(t)$.

Let us also assume that the technology of the leader (the *frontier technology*) grows with a constant exogenous rate g:

$$T(t) = T(0)e^{gt}. \tag{1.74}$$

In other countries, in which the level of technology is lower, technology development can result from both research and development (R&D) activity and technology diffusion from the leader country. The diffusion effect can be modelled in two ways – with the so-called *confined exponential diffusion* (the Nelson-Phelps model [61]) or the *logistic model* (Benhabib and Spiegel [16]).

The growth rate of technology in the model with confined exponential diffusion equals in the i-th country:

$$\frac{\dot{A}_i(t)}{A_i(t)} = g\left(h_i(t)\right) + c\left(h_i(t)\right)\left(\frac{T(t)}{A_i(t)} - 1\right), \tag{1.75}$$

where $g\left(h_i(t)\right)$ denotes an increasing function of human capital $h_i(t)$, which describes the dependence of the innovation rate (efficiency of the R&D activity) on the level of human capital in the i-th country and $c\left(h_i(t)\right)$ denotes an increasing function of human capital, which reflects the dependence of the rate of diffusion on the level of human capital in this country. The original Nelson-Phelps model is reproduced if $g \equiv 0$ and the function $c\left(h(t)\right)$ satisfies the condition $c(0) = 0$.

In the logistic model, the growth rate of technology equals:

$$\frac{\dot{A}_i(t)}{A_i(t)} = g\left(h_i(t)\right) + c\left(h_i(t)\right)\frac{A_i(t)}{T(t)}\left(\frac{T(t)}{A_i(t)} - 1\right). \tag{1.76}$$

In the model given by (1.75), the more technologically underdeveloped a country (low A_i) and the higher the human capital endowment (high h_i), the higher the diffusion rate. If the level of human capital is constant over time, one can give the analytical form of the solution to equation (1.75):

$$A_i(t) = \left(A_i(0) - \Omega T(0)\right)e^{(g_i - c_i)t} + \Omega T(0)e^{gt}, \tag{1.77}$$

where g_i and c_i denote, respectively, $g\left(h_i(t)\right)$ and $c\left(h_i(t)\right)$ in the case when h_i does not depend on time, and:

$$\Omega = \frac{c_i}{c_i - g_i + g}. \tag{1.78}$$

One can also show that:

$$\lim_{t\to\infty}\frac{A_i(t)}{T(t)} = \Omega, \tag{1.79}$$

i.e. regardless of the parameter values c_i, g_i and g, in the limit $t \to \infty$ technology grows at the same rate in all countries and there emerges some distribution of A_i in the world. If $g_i = g$, then the i-th country will at some time catch up with the technological leader ($\Omega = 1$).

Introducing another factor in the logistic model equation suppresses the diffusion rate if the underdevelopment of some country is excessive, reflecting difficulties in adopting too modern technologies in such country.

The analytical solution to equation (1.76), if human capital is time-independent, takes the form:

$$A_i(t) = \frac{A_i(0)e^{(g_i + c_i)t}}{1 + \frac{A_i(0)}{T(0)}\frac{c_i}{c_i + g_i - g}\left(e^{(c_i + g_i - g)t} - 1\right)}. \tag{1.80}$$

One can show that:

$$\lim_{t\to\infty}\frac{A_i(t)}{T(t)}=\begin{cases}(c_i+g_i-g)/c_i, & \text{when} \quad c_i+g_i-g>0\\ A_i(0)/T(0), & \text{when} \quad c_i+g_i-g=0\\ 0, & \text{when} \quad c_i+g_i-g<0\end{cases}, \tag{1.81}$$

i.e. the ratio of i-th country's technology to the frontier technology in the limit $t\to\infty$ depends on the relationship of the rate of diffusion c_i and the difference in the innovation rates g_i-g. If the innovation rate in the i-th country is too low, then this country will never catch up with the leader and its technology will be constant or declining relatively to the leader.

The vital difference between the logistic and the confined exponential model consists in the fact that in the latter the underdeveloped country always chases the leader, while in the former it is possible that some countries will not be able to keep pace with the growth of the frontier technology and their technological underdevelopment will thus grow over time.

The key role in both models is played by human capital, which is the source of diffusion effects. Human capital investments always increase the diffusion rate and in the case of technological divergence are able to reverse the trend. Hence, there exists some minimal level of human capital that makes possible technological convergence (this is an example of a threshold externality, as discussed in the context of the model of Azariadis and Drazen [8]).

Empirical research conducted by Benhabib and Spiegel showed that the logistic model is more realistic, i.e. technological divergence is possible in the countries with lowest human capital resources, in which the combined diffusion and innovation rates are lower than the growth rate of technology of the leader.

Another work, closely related to the above discussion, was the one of Phelps' of 1966 [64]. The author considered a closed economy with Harrod-neutral technological progress. The rate of technology growth depended in this model on the quantity of *effective research* and on the level of technology from a few periods before, reflecting the lag between innovation and its practical application. Phelps found an expression for the optimal savings rate, resulting from the optimal division of physical capital and labour between the productive and the technological sector.

In 1970 Gomulka [30] introduced three generalizations of Phelps' approach for an open economy. In the first, technological progress was possible because of innovations import. Gomulka derived an expression for optimal savings rate and showed that if some conditions hold, then technological convergence is possible. In the second variant, technological sector was divided into two branches – experimental and theoretical. The level of theoretical research in a country determined the growth perspectives in the experimental sector, which in turn influenced the production sector. The author showed what is the optimal division of labour and capital between the research and productive sectors, to maximize consumption at some given time. In the third variant, Gomulka considered the division of the technological sector into infinitely many levels of research, such that any level influenced the efficiency of research in the higher levels[11]. Also for this case, the author showed the optimal allocation of resources between the sectors.

[11] The second variant is actually a particular case of the third one with only two levels of research.

1.3.2 The Romer model

The Romer model [68] of 1990 is one of the first models of endogenous technological progress[12]. The development of technology results in this model from rational decisions of agents who aim at maximizationof their own profit.

There are four factors of production in the model – physical capital K, labour L and knowledge-related human capital H and technology A. A vital difference between these factors is the fact that using human capital in one sort of activity makes it impossible to use it in any other activity. When it comes to technology, it can be used simultaneously in many kinds of activities.

One can distinguish three sectors in the modelled economy. The research sector uses human capital and existing knowledge to "produce" new knowledge in the form of technologies of production of intermediate goods. The intermediate goods sector, in turn, uses technology and physical capital and supplies its products to the final goods sector, which uses also labour and human capital.

To simplify, one assumes that labour L is constant. The resource of human capital of the whole economy H, as well as its division between the research and final goods sectors is also fixed.

The physical capital dynamics equation takes the standard form:

$$\dot{K}(t) = Y(t) - C(t), \tag{1.82}$$

where $C(t)$ denotes aggregative consumption at time t and $Y(t)$ the final goods production at this moment

Human capital can be used in the research sector (H_A) or in the final goods sector (H_Y) in such a way that:

$$H = H_A + H_Y. \tag{1.83}$$

We assume that the production function takes the form:

$$Y(H_Y, L, x) = H_Y^{\alpha} L^{\beta} \int_0^{\infty} x(i)^{1-\alpha-\beta} di, \tag{1.84}$$

where $x(i)$ denotes the quantity of i-th good that is used in the production of the final good. It is possible to use only these intermediate goods, for which the technology of production is known. Let us denote the most technologically advanced intermediate good by A and let us assume that $\forall\, i > A, \quad x(i) = 0$. The variable A can be thus interpreted as the technology level of the economy. A unit of physical capital comes from η units of each of intermediate goods:

$$K = \eta \int_0^{A} x(i) di. \tag{1.85}$$

We assume that each intermediate good is produced by a single firm, who paid the price P_A for an exclusive patent. The prices of intermediate goods are given by the function $p(i)$, which can be shown to depend on the aggregative demand in the following way:

$$p(i) = (1 - \alpha - \beta) H_Y^{\alpha} L^{\beta} x(i)^{-\alpha-\beta}. \tag{1.86}$$

[12]The first endogenous technological progress model is often considered to be another model by Romer [67] of 1986. This model is, however, equivalent to the Arrow model [5] of 1962.

One can also show that in equilibrium the profit of the intermediate good producer $\pi(t)$:

$$\pi(t) = r(t)P_A, \tag{1.87}$$

where $r(t)$ denotes the interest rate at time t.

We also assume the following form of the knowledge accumulation equation:

$$\dot{A}(t) = \delta H_A A, \tag{1.88}$$

where δ is the parameter that determines the maximal growth rate of technology. The rate of intermediate goods technology production depends in this way on the quantity of research human capital H_A and on total knowledge. The assumption of linear dependence on A is vital to obtain a model with long-run economic growth, impossible with a strictly concave function of A in equation (1.88).

The consumer preferences are given similarly as in the Ramsey-Cass-Koopmans model, which implies the following dynamics equation for the function $C(t)$, which holds if r is constant (which we assume from now on):

$$\frac{\dot{C}(t)}{C(t)} = \frac{r-\rho}{\theta}, \tag{1.89}$$

where ρ is the discount rate and θ the reverse of elasticity of substitution between consumption in two arbitrary moments. The consumers maximize their utility by choosing a fraction of their human capital that they will supply to the research and the final goods sectors. If we denote wage per unit of human capital by w_H, then the knowledge accumulation equation (1.88) leads to the following relation:

$$w_H = P_A \delta A. \tag{1.90}$$

The symmetry of the model implies that all available intermediate goods are used in the same amount, denoted by $\bar{x}$ (constant over time if r is constant). The production function (1.84) can thus be simplified to:

$$Y(H_Y, L, x) = H_Y^{\alpha} L^{\beta} A \bar{x}^{1-\alpha-\beta}. \tag{1.91}$$

Using the relation between physical capital and technology in the case of a constant function $x(i)$, we have:

$$K = \eta A \bar{x}, \tag{1.92}$$

hence we obtain:

$$Y(H_Y, L, x) = (H_Y A)^{\alpha} (LA)^{\beta} K^{1-\alpha-\beta} \eta^{\alpha+\beta-1}. \tag{1.93}$$

The production function (1.93) is a neoclassical production function of Cobb-Douglas type with labour and human capital oriented technological progress.

Such form of the production function implies that we can find a steady-state, in which the key variables of the model grow at the same rate.

Equilibrium at the human capital market takes place if the wages per unit of research human capital and final goods human capital are equal to marginal productivities of human capital in these sectors, i.e.:

$$P_A \delta A = \alpha H_Y^{\alpha-1} L^{\beta} A \bar{x}^{1-\alpha-\beta}, \tag{1.94}$$

where the right-hand side describes the marginal productivity of human capital in the final goods sector and results from equation (1.91). Taking into account expressions (1.86), (1.87) and $H_A = H - H_Y$, we obtain an equation for the fraction of human capital in the research sector:

$$H_A = H - \frac{\Lambda}{\delta} r, \tag{1.95}$$

where:

$$\Lambda = \frac{\alpha}{(1-\alpha-\beta)(\alpha+\beta)}. \tag{1.96}$$

The steady state exists if H_Y and H_A are constant in time. It results from equation (1.95) that this is possible only if the interest rate does not change, which we assumed before.

The growth rate of technology in the steady state $\dot{A}/A$ equals δH_A in accordance with (1.88). It results then from equations (1.92), (1.91) and (1.82) that also the physical capital, final goods production and consumption grow at the same rate, which we denote g. We can thus write:

$$g = \frac{\dot{C}}{C} = \frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = \frac{\dot{A}}{A} = \delta H_A. \tag{1.97}$$

We obtain from equation (1.95):

$$g = \delta H - \Lambda r, \tag{1.98}$$

and using the relation (1.89) between the interest rate and the discount rate, we obtain finally:

$$g = \frac{\delta H - \Lambda\rho}{\theta\Lambda + 1}. \tag{1.99}$$

The rate of technological progress in the Romer model grows if the level of human capital increases in the economy and if the average productivity of this capital grows in the research sector (the parameter δ). Technology growth rate, however, decreases, when the discount rate increases – for high values of ρ it is not profitable to invest in future consumption by allocating human capital in research.

The fraction of research human capital, calculated above, is too low from the social point of view. The representative consumer who maximizes their utility does not take innovation externalities into account – every new technology of intermediate goods production increases the efficiency of every researcher, which is not reflected in the price of this technology and the innovator's profit.

The *socially optimal allocation* can be found if we consider the following utility maximization problem of the whole society:

$$\max \int_0^\infty u\left(C(t)\right) e^{-\rho t} dt, \tag{1.100}$$

in which the utility function is given by (1.39), subject to (1.83), (1.88) and (1.82) with the production function (1.93). One can show that the socially optimal steady-state, which results from the optimality conditions for the above problem, is characterized by the following expression for the fraction of human capital allocated to research:

$$H_A^* = \frac{\frac{\delta(\alpha+\beta)}{\alpha} H - \rho}{\delta\left(\theta - \frac{\beta}{\alpha}\right)}. \tag{1.101}$$

The growth rate for the key variables g^* equals:

$$g^* = \frac{\delta H - \Theta\rho}{\Theta\theta + 1 - \Theta}, \tag{1.102}$$

where:

$$\Theta = \frac{\alpha}{\alpha+\beta}. \tag{1.103}$$

Since $\Theta < \Lambda$, the socially optimal growth rate is higher than g and the socially optimal fraction of human capital employed in the research sector H_A^* exceeds H_A.

The key property of the research process in the Romer model is a twofold influence of knowledge on the economy. Firstly, the limits of knowledge determine the range of intermediate goods, which in turn influences the efficiency of final goods production. Secondly, knowledge determines also the productivity of human capital in the research sector, allowing for faster technological progress. In this way, *knowledge is the essential factor of economic growth.*

The Romer model was one of the first models of endogenous growth and, similarly to the Solow-Swan model, the ideas underlying this model were the basis for many further models of the so-called *horizontal technological progress* – progress related to a growing number of available producer or consumer goods.

1.3.3 The Jones model

The Jones model is a human capital and technological progress based model of economic growth. The mechanism for the development of technology is essentially the same as in the Romer model and for human capital investments it is similar to the Lucas model.

There are three kinds of goods in the economy under consideration – consumer (Y), intermediate ($x(i)$) – termed *ideas* and human capital (h) – which determines the range of intermediate goods that can be used within a given firm. The total amount of labour L is divided exogenously between the production of the three types of goods, i.e.:

$$L = L_Y + L_A + L_h, \tag{1.104}$$

where L_Y denotes the labour resource used in consumer goods production, L_A in ideas "production" and L_h in human capital accumulation. The consumer goods are produced with labour L_Y and a set of intermediate capital goods $x(i)$.

The production function for a representative firm which employs workers endowed with an average human capital h is given by:

$$Y(t) = L_Y(t)^{1-\alpha} \int_0^{h(t)} x(i,t)^{\alpha} di, \tag{1.105}$$

where $\alpha \in (0,1)$ is a parameter.

The human capital dynamics equation takes the form:

$$\dot{h}(t) = \mu e^{\theta u(t)} h(t) \left(\frac{A(t)}{h(t)} \right)^{\gamma}, \tag{1.106}$$

where μ, θ are positive parameters, γ is a non-negative parameter, $u(t)$ denotes the exogenously given fraction of time allocated to human capital accumulation (hence, $L_h = uL$) and $A(t)$ represents the technological frontier, i.e. the range of intermediate goods (ideas), which can be "produced" in the economy. Equation (1.106) is a generalized version of the respective equation from the Lucas model[13]. The case $\gamma > 0$ corresponds to the situation, when it is easier to learn to use the less advanced intermediate goods than the more advanced ones.

[13]In the Lucas model one has $\gamma = 0$ (the growth rate of human capital $\dot{h}/h$ is thus h-independent) and the fraction of time allocated to human capital accumulation has the form $u(t)$, and not $e^{\theta u(t)}$.

1

The ideas production function is given by:

$$\dot{A}(t) = \delta h(t)^{\beta} L_A(t) A(t)^{\phi}, \tag{1.107}$$

where δ, β and ϕ are positive parameters. This equation is a modified Romer's equation (1.88), in which the dependence between $\dot{A}$ and A is not linear. The factor A^{ϕ} reflects the influence of the earlier ideas on current research. A similar effect is exerted by the factor h^{β}, which is, however, related to the individuals' abilities (their education and experience), and not to general knowledge of a society, represented by the variable A.

Physical capital is accumulated according to the equation:

$$\dot{K}(t) = sY(t) - dK(t), \tag{1.108}$$

where s denotes the exogenously given fraction of production devoted to physical capital investments (the fraction $(1-s)$ is consumed) and d is the physical capital depreciation rate. Intermediate goods are produced from physical capital:

$$\int_0^{h(t)} x(i,t)di = K(t). \tag{1.109}$$

Similarly to the Romer model, intermediate goods are symmetrical, i.e. $\forall i \;\; x(i,t) = x(t)$. It implies the production function of the following form:

$$Y(t) = K(t)^{\alpha} \left(hL_Y\right)^{1-\alpha}. \tag{1.110}$$

Let us now consider the steady-state in the Jones model. We divide equation (1.107) by $A(t)$ and obtain:

$$\frac{\dot{A}(t)}{A(t)} = \delta \left(\frac{h(t)}{A(t)}\right)^{\beta} \frac{L_A(t)}{A(t)^{1-\beta-\phi}}. \tag{1.111}$$

In the steady-state A and h grow at the same rate. Hence, the growth rate of the technological frontier $\dot{A}/A \equiv g_A$ equals:

$$g_A = \frac{n}{1-\beta-\phi}. \tag{1.112}$$

The steady-state exists only if $\beta + \phi < 1$. One can show that the remaining key variables of the Jones model grow at the rate g_A:

$$g_y = g_k = g_h = g_A \equiv g, \tag{1.113}$$

where g_x is the growth rate of the variable x; $y \equiv Y/L_Y$, $k \equiv K/L_Y$. One can also obtain an equation for production per worker in the consumer goods sector:

$$y^*(t) = \left(\frac{s}{n+g+d}\right)^{\frac{\alpha}{1-\alpha}} \left(\frac{\mu}{g} e^{\theta u}\right)^{\frac{1}{\gamma}} A^*(t). \tag{1.114}$$

This equation is analogous to equations (1.23) and (1.73) and makes it possible to conduct empirical research based on this model.

To sum it up, there are essentially two different kinds of accumulated knowledge in the Jones model – a set of ideas, which are used in the production of consumer goods and more advanced ideas, and the human capital resource, which can be interpreted as the ability to use the ideas in the productive processes. To simplify, Jones imposed an exogenous mechanism of allocation of production factors. However, he has also introduced models with endogenous allocation mechanisms (Jones [42], Jones [44], Jones and Williams [45], Jones [46]). Models of this kind make it possible to look more comprehensively at the economic role of knowledge and seem to be the most appropriate way to consider economic growth – therefore their more careful examination and possible extensions can contribute much to a better understanding of economic growth.

1.3.4 Other models

In this subsection we will shortly review alternative models of technological progress.

A very important branch of technological progress theory was started by Grossman and Helpman [33], [34], [35] in 1991. The so-called *vertical technological progress* is embodied in an improving quality of available goods. Each product can be placed somewhere on a quality ladder and higher quality corresponds to a higher utility of consumption of a unit of this product. The move of a product up the quality ladder is modelled as a stochastic process, in which the probability of a research success (which leads to a quality improvement) depends on the expenses allocated to research. In the papers [34] and [35], the quality ladder model is considered in the context of international trade and technology diffusion. The world economy can be divided into two areas – North and South[14]. Quality-improving research is conducted only in the North. Traders in the South do not have enough resources and abilities to conduct research – thus the only way to develop their technologies is by imitating the ever-improving products from the North. In the steady-state there is an equilibrium between innovation and imitation processes.

In 1992, Aghion and Howitt [4], independently from Grossman and Helpman, proposed a similar model. They considered vertical technological progress, which consists in an improvement in quality of intermediate goods. The authors emphasized the meaning of *creative destruction* for technological progress and the whole economy – inspired by Schumpeter's ideas from the first half of the twentieth century.

In 2002 Acemoglu [2] introduced a model of directed technological change. In the nineteenth century, technological change was directed at unqualified and a century later at highly-qualified labour[15] (especially after 1980). In subsection 1.1.3 we have distinguished Solow, Harrod and Hicks-neutral technological progress. If we consider production functions other than Cobb-Douglas, technological progress has to be Harrod-neutral to guarantee the existence of a steady-state. The Acemoglu model makes it possible to explain the fact that technological change is labour-oriented – thus Harrod-neutrality does not have to be an assumption of the model, but can result from the model itself.

Another problem which is addressed within this branch of economics is the one of appropriate technologies. In 1969, Atkinson and Stiglitz [6] suggested that every production technology can be appropriate only if the capital-labour ratio is adequate. If one tries to use too advanced technologies in a country with a low capital-labour ratio, the output will be lower than in the case when a more adequate technology is applied. This idea was developed in 1998 by Basu and Weil [12], who introduced a dynamical version

[14]This refers to the Krugman model [51] of 1979, in which it was shown that international trade can lead to the diffusion of technology between the rich North and the poor South.

[15]Such effects can be observed by analyzing supply of people with higher education and their real wages. Despite an enormous growth of supply of such workers, their wages grew quickly. If technical change had not been high qualified labour oriented, one would have

of the Atkinson-Stiglitz model. Also, Acemoglu and Zilibotti [1] assumed in 2001 that a given technology of production is adequate if human capital per worker is high enough, i.e. if the workers are appropriately qualified. Too low human capital resources are, in the view of Acemoglu and Zilibotti, the cause of low efficiency of technological diffusion processes to the least developed countries.

To conclude this section, let us enumerate a few empirical papers on technological progress. One usually analyzes the relationship between R&D expenses, the number of patents and economic growth. The most important papers were written by Griliches (e.g. [31], [32]), Keely (e.g. [48]), Oakes (e.g. [63]), Hall (e.g. [36]), Evenson (e.g. [29]) and Kortum (e.g. [50]). Another strand of research emphasizes the role of academic performance indicators, e.g. Adams [3] and Dasgupta with David [27]. Widely analyzed is also the phenomenon of knowledge and technology diffusion at the microeconomic level (e.g. Jaffe [40], Ciccone and Hall [22], Jaffe, Trajtenberg and Henderson [41]) or at the macroeconomic level (e.g. Coe, Helpman and Hoffmeister [25] (based on the Grossman-Helpman models), Eaton and Kortum [28]).

Chapter 2

The Manuelli-Seshadri model

In this chapter we will analyze an advanced human capital model of Manuelli and Seshadri [59] of 2005. In particular, we will show the setup of the model, its solution, steady-state properties and results of empirical analysis for the OECD countries.

2.1 Setup of the model

The Manuelli-Seshadri model differs significantly from human capital models analyzed in section 1.2. The most important difference is that the notion of the *quality* of human capital is employed. In this way, one can take into account differences in education quality in different countries.

Let us consider an economy, in which the representative individual is born when their parents are $B = 25$ years old. They go to school at age 6 for s years. During this time, the individual devotes all of their time to human capital accumulation. At age $6 + s$, they start to work, but further invest in human capital. When the individual is $R = \min\{64, T\}$ years old, where T is their lifespan, they retire.

Human capital accumulation is modelled in a similar way to the Ben-Porath model [17]. The representative individual chooses such growth paths of human capital $h(a)$, where a denotes their age, time allocated to human capital accumulation $n(a)$ and expenditures on market goods related to human capital accumulation $x(a)$, such that their discounted earnings from their whole life are maximal. The income maximization problem of the representative individual reads:

$$\max \int_6^R e^{-r(a-6)} \left(wh(a)(1-n(a)) - x(a)\right) da - x_E, \qquad (2.1)$$

subject to the human capital dynamics equation:

$$\dot{h}(a) = z_h \left(n(a)h(a)\right)^{\gamma_1} x(a)^{\gamma_2} - \delta_h h(a), \qquad a \in [6, R), \qquad (2.2)$$

and the "technology" of early human capital production $h(6) \equiv h_E$ of the form:

$$h(6) \equiv h_E = h_B x_E^{\nu}, \qquad (2.3)$$

where w denotes the wage rate per unit of human capital, r the discount rate, x_E the parents' expenditures during early childhood of their children (until they are 6 years old), δ_h the human capital depreciation rate and z_h, γ_1, γ_2, h_B and ν are positive parameters.

The solution to the problem (2.1)-(2.3) by optimal control techniques is given in Appendix A.

2

2.2 Optimal growth paths of human capital and the time and expenditures related to its accumulation

The solution to the income maximization problem (2.1)-(2.3) is given by the following growth paths of human capital.

The early childhood human capital (at age 6) equals:

$$h_E = \left(h_B v^v \left(\frac{w^{(1-\gamma_1)(1-\gamma_2)} \gamma_1^{\gamma_1(1-\gamma_2)} \gamma_2^{\gamma_1\gamma_2} z_h^{\gamma_1}}{(r+\delta_h)^{1-\gamma_2} m(6+s)^{\gamma_2-1}} \right)^{\frac{v}{1-\gamma}} e^{-v(r+\delta_h(1-\gamma_1))s} \right)^{\frac{1}{1-v(1-\gamma_1)}}, \tag{2.4}$$

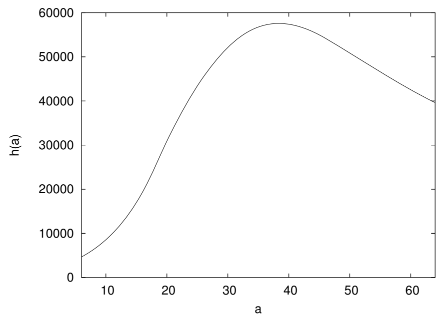
where $\gamma = \gamma_1 + \gamma_2$ and

$$m(a) = \left(1 - e^{-(r+\delta_h)(R-a)}\right). \tag{2.5}$$

During formal education ($a \in [6, 6+s]$), human capital grows as:

$$\begin{aligned} h(a) &= h_E e^{-\delta_h(a-6)} \left(1 + \left(h_E^{-(1-\gamma)} q_E^{\gamma_2} \gamma_2^{\gamma_2} z_h \right)^{\frac{1}{1-\gamma_2}} \times \right. \\ &\times \left. \frac{(1-\gamma_1)(1-\gamma_2)}{\gamma_2 r + \delta_h(1-\gamma_1)} \left(e^{\frac{\gamma_2 r + \delta_h(1-\gamma_1)}{1-\gamma_2}(a-6)} - 1 \right) \right)^{\frac{1}{1-\gamma_1}}, \end{aligned} \tag{2.6}$$

Figure 2.1: **An example of a human capital growth path (in arbitrary units) of a representative individual of age 6 to 64**



where:

$$q_E = \frac{e^{-(1-v)(r+\delta_h(1-\gamma_1))s}}{h_B v^v} \left(\frac{w^{(1-\gamma_1)(1-\gamma_2)} \gamma_1^{\gamma_1(1-\gamma_2)} \gamma_2^{\gamma_1\gamma_2} z_h^{\gamma_1} m(6+s)^{1-\gamma_2}}{(r+\delta_h)^{1-\gamma_2}} \right)^{\frac{1-v}{1-\gamma}}. \tag{2.7}$$

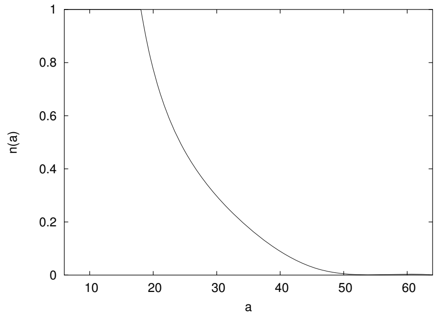
When the age of the representative individual is $a \in [6+s, R]$, we have:

$$\begin{aligned} h(a) &= e^{-\delta_h(a-6-s)} \left(h(6+s) + \frac{1}{\delta_h} \left(\frac{\gamma_2^{\gamma_2} \gamma_1^{\gamma_1} z_h w^{\gamma_2}}{(r+\delta_h)^{\gamma}} \right)^{\frac{1}{1-\gamma}} \times \right. \\ &\times \left. e^{-\delta_h(6+s-R)} \int_{e^{\delta_h(6+s-R)}}^{e^{\delta_h(a-R)}} dx \left(1 - x^{\frac{r+\delta_h}{\delta_h}} \right)^{\frac{\gamma}{1-\gamma}} \right), \end{aligned} \tag{2.8}$$

where $h(6+s)$ is given by equation (2.6).

An example of the dynamics of human capital of the representative individual is given in Fig. 2.1. It corresponds to the parameters for the American economy ca. the year 2000. In particular, the average length of schooling for the representative individual equals $s = 12.08$ years. In the schooling period, human capital exhibits fast growth and after this period it slows down (the point $6+s$ is always the inflection point of the curve $h(a)$), and at some age, close to the half of the individual's earning period, human capital depreciation becomes larger than its production and the level of human capital shrinks until retirement at age R.

Figure 2.2: **An example of the dynamics of the fraction of time devoted to human capital accumulation of a representative individual of age 6 to 64**



The time devoted to human capital accumulation fulfills during the schooling period the condition:

$$n(a) = 1, \tag{2.9}$$

and in the working period:

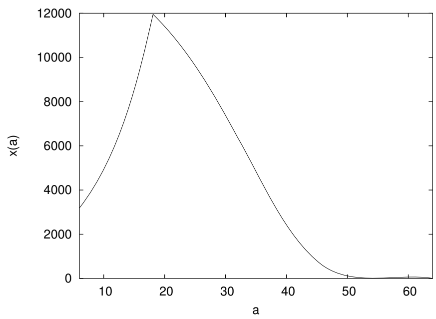
$$n(a) = \frac{m(a)^{\frac{1}{1-\gamma}}}{e^{-\delta_h(a-s-6)}m(6+s)^{\frac{1}{1-\gamma}} + \frac{(r+\delta_h)e^{\delta_h(R-a)}}{\gamma_1\delta_h}\int_{x_{6+s}}^{x_a} dx \left(1 - x^{\frac{r+\delta_h}{\delta_h}}\right)^{\frac{\gamma}{1-\gamma}}}, \tag{2.10}$$

where $x_a = e^{\delta_h(a-R)}$, $x_{6+s} = e^{\delta_h(6+s-R)}$.

An example of the fraction of time devoted to human capital accumulation of the representative individual aged 6 to 64 is given in Fig. 2.2. During the schooling period, the individual does not work and when this period is over, the fraction of time allocated to human capital production is a decreasing function of age. At some age it reaches zero and is zero until retirement – it is not profitable for the individual to give up a part of their earnings, since the time until retirement is too short.

Expenditures for market goods allocated to human capital accumulation are given by the following equations:

Figure 2.3: **An example of the dynamics of expenditures on market goods allocated to human capital accumulation of a representative individual of age 6 to 64**



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- during the schooling period:

$$x(a) = (q_E h_E^{\gamma_1} \gamma_2 z_h)^{\frac{1}{1-\gamma_2}} e^{\frac{r+\delta_h(1-\gamma_1)}{1-\gamma_2}(a-6)}, \tag{2.11}$$

- during the working period:

$$x(a) = \frac{\gamma_2 w}{r+\delta_h} \left(\frac{\gamma_2^{\gamma_2} \gamma_1^{\gamma_1} z_h w^{\gamma_2}}{(r+\delta_h)^{\gamma}} \right)^{\frac{1}{1-\gamma}} m(a)^{\frac{1}{1-\gamma}}. \tag{2.12}$$

Fig. 2.3 shows an example of the dynamics of expenditures on market goods allocated to human capital accumulation of a representative individual aged 6 to 64 years. During the schooling period, the expenditures grow to reach a maximum at the end of the period. Afterwards, they begin to decrease to reach zero at the age when human capital investments are not profitable any more.

2.3 Per capita variables

To apply the model to real economies we need to take their demographic structure into account.

We assume that the representative individual has e^f children (the growth rate of population equals then $\eta = f/B$), born at age $B = 25$ years. The age structure of the population is given by the equations:

$$N(a,t) = \phi(a) e^{\eta t}, \tag{2.13}$$

where:

$$\phi(a) = \eta \frac{e^{-\eta a}}{1 - e^{-\eta T}}. \tag{2.14}$$

The number of people of age between a and $a + \delta a$ at time t equals thus $\int_a^{a+\delta a} N(a', t) da'$. If $\eta = 0$, then the de l'Hôpital's theorem implies that expression (2.14) simplifies to:

$$\phi(a) = 1/T, \tag{2.15}$$

This means that the number of people in any given interval of the same width is the same.

If we know the age structure of the economy, we can calculate the average human capital p.c. ($1 - n(a)$ denotes the time allocated to work by the representative individual of age a):

$$\bar{h} = \frac{\int_{6+s}^{R} h(a)\,(1 - n(a))\,\phi(a) da}{\int_{6+s}^{R} \phi(a) da}. \tag{2.16}$$

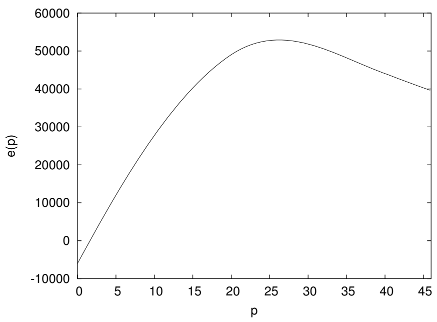
In empirical research, we will often use the average earnings per worker. The dependence of earnings $e(p)$ on experience $p \equiv a - 6 - s$ can be expressed as the difference between the income from work and expenditures on education:

$$e(p) = wh(p+6+s)(1 - n(p+6+s)) - \pi x(p+6+s), \tag{2.17}$$

where π denotes the fraction of expenditures on market goods allocated to human capital accumulation that are paid by the employee (the fraction $1-\pi$ is paid by the employer; we will take $\pi = 0.5$). Substituting the expressions for $h(a)$, $n(a)$ and $x(a)$ at age $a = p + 6 + s$, we obtain:

$$e(p) = \left(\frac{\gamma_2^{\gamma_2}\gamma_1^{\gamma_1} z_h w^{\gamma_2}}{(r+\delta_h)^{\gamma}}\right)^{\frac{1}{1-\gamma}} w\left(\gamma_1 e^{-\delta_h p}\frac{m(6+s)^{\frac{1}{1-\gamma}}}{r+\delta_h} - (\gamma_1 + \pi\gamma_2)\right. \quad (2.18)$$
$$\left.\times\frac{m(p+6+s)^{\frac{1}{1-\gamma}}}{r+\delta_h} + \frac{e^{-\delta_h(p+6+s-R)}}{\delta_h}\int_{e^{\delta_h(6+s-R)}}^{e^{\delta_h(p+6+s-R)}} dx\left(1 - x^{\frac{r+\delta_h}{\delta_h}}\right)^{\frac{\gamma}{1-\gamma}}\right).$$

Figure 2.4: **An example of the dependence between the earnings and experience**



An example of the dependence of earnings on experience is shown in Fig. 2.4. Initially, the education expenditures are higher than the earnings from work ($n(a)$ is close to 1) and the net income is negative. Along with the growing experience, the level of human capital $h(a)$ increases and the resources allocated to human capital accumulation ($n(a)$ and $x(a)$) decrease, which results in the growth of earnings, which are maximal around the half of the working life and then start to shrink, which results from human capital depreciation.

The average wage in the economy (denoted E) can be obtained in an analogous way to the average human capital p.c.:

$$E = \frac{\int_{6+s}^{R} e(a-6-s)\phi(a)da}{\int_{6+s}^{R}\phi(a)da}. \quad (2.19)$$

2.4 Equilibrium

The Manuelli-Seshadri model describes only one branch of economy – the human capital accumulation sector. However, one can supplement the model by introducing the production function $F(k, h)$ and the equations that express the equilibrium conditions on the physical capital (k) and human capital market (h).

We assume that the production function is homogeneous of degree one. The physical capital market equilibrium takes place when the marginal cost and product of physical capital are equal to each other:

$$p_k(r + \delta_k) = \frac{\partial F(\kappa, 1)}{\partial k}, \quad (2.20)$$

where p_k denotes the relative price of physical capital with respect to some reference country, r is the interest rate, δ_k the physical capital depreciation rate, $\kappa \equiv k/h$.

Analogously, equilibrium on the human capital market means that the marginal product of human capital equals the wage rate per unit of human capital:

$$w = \frac{\partial F(\kappa, 1)}{\partial h}. \tag{2.21}$$

Gross domestic product (GDP) per worker can be expressed by the average human capital p.c. $\bar{h}$:

$$y = F(\kappa, 1)\bar{h}. \tag{2.22}$$

The above equations are the basis for empirical research that uses the Manuelli-Seshadri model.

2.5 Empirical applications

2.5.1 The calibration of the model

The model was calibrated by the authors to describe the American economy ca. 2000. They used the Cobb-Douglas production function:

$$F(k, h) = Ak^{\theta}h^{1-\theta}, \tag{2.23}$$

where A denotes the index of technology, commonly referred to as TFP – Total Factor Productivity. Thus, one obtains the following equation for GDP p.c.:

$$y = A\kappa^{\theta}\bar{h}. \tag{2.24}$$

The physical capital depreciation rate is taken at a standard level of $\delta_k = 6\%$ and the interest rate at $r = 7\%$. The demographic parameters are $B = 25$, $R = \min\{64, T\}$ and $f = 0$ (for other countries we take f at their actual levels). For USA, the reference country, we take $A = 1$, $p_k = 1$ and $w = 1$.

Table 2.1: **The calibrated values of the Manuelli-Seshadri model parameters**

Parameter	δ_k	δ_h	z_h	θ	γ_1	γ_2	v	r
value	0.06	0.018	0.361	0.315	0.63	0.30	0.55	0.07

Source: Manuelli and Seshadri [59].

The remaining 7 parameters of the model are taken at such values that the solution of the model is consistent with the following empirical data for the American economy:

- capital's share of income of 0.33,
- capital output ratio of 2.52,
- earnings at age R to earnings at age 55 of 0.8,
- earnings at age 50 to earnings at 25 of 2.17,
- years of schooling of $s = 12.08$,
- schooling expenditures per pupil relative to GDP p.c. of 3.77%,
- pre-primary expenditures per pupil relative to GDP p.c. of 14%.

In this way, we obtain the parameter values given in Tab. 2.1. The values of these parameters are the same for all countries. Thus, we do not assume a priori differences in the ability to learn (parameter z_h) and the human capital depreciation rate (parameter δ_h). The value of the parameter h_B depends in the model on the assumed values of A and p_k – thus it can be different for different countries.

Manuelli and Seshadri applied their model in the following way. They took actual demographic data for each country and chose such value of the technology index A to yield GDP p.c. (eq. (2.24)) consistent with the empirical value. Thus, the model allowed them to find the average years of schooling and the expenditures allocated to human capital accumulation of the representative individual of a given country.

The model, however, can also be used to search for the dependence between human capital and national income. If we introduce the actual values of the years of schooling of the representative individual, the average wage rate per unit of human capital and the relative price of physical capital, we can obtain estimates of the average level of human capital and gross domestic product in the analyzed country.

2.5.2 Methods of estimation of the wage rate per unit of human capital

The key variable in the Manuelli-Seshadri model is the wage rate per unit of human capital w. In empirical research we will use the following strategy regarding w.

In the next subsection we will perform a sensitivity analysis with respect to w for Poland. In particular, we will find the *empirical* value of w, i.e. such value that leads to the actual value of GDP p.c. and we will examine its implications.

Next, we will rest on the empirical results of Hendricks [38], concerning the average immigrants' wage of people from various countries on the American labour market. We will continue in the following way. From eq. (2.19) we will calculate the average wage of the American with a nominal education (given by some given years of schooling s) corresponding to the average immigrant from a given country. We will determine the value of w which leads to the empirical immigrant's wage (with respect to the American of the same nominal education), given by Hendricks. In this way, the parameter w will let us to isolate the influence of the quality of human capital on immigrants' wages. Therefore, the differences in human capital resources can result from different lengths of schooling (the parameter s) and from differences in the qualities of this process (the parameter w). Our analysis is based on the assumption that the American labour market correctly assesses the abilities (the human capital resources) of immigrants from various countries. However, this assumption does not always hold – we can imagine that the wages of some immigrants can be higher than the ones resulting from their abilities due to socio-political reasons, since too low a wage could result in accusations of abusing the immigrants and treating them as a cheap workforce. In other words, the immigrants' human capital can be of a much worse quality than Americans' human capital with the same schooling period, but the difference in wages sometimes might not reflect this difference.

Therefore, we will also use another method of estimation of the wage rate per unit of human capital, using the results of the *International Adult Literacy Survey* (IALS), i.e. a 22-country survey of adult literacy. Three aspects were investigated: reading comprehension, document comprehension and simple calculations. Coulombe *et al.* [26] found that a one-percent increase in the average IALS result leads to a 2.5-percent increase in productivity of a worker from this country. Hence, if we know the average IALS result (with respect to USA), then we also know the ratio of productivities of the workers from a given country and from USA. We can thus assume that the ratio of incomes equals the ratio of productivities and we can find such value of w that leads in the Manuelli-Seshadri to this ratio of incomes (from eq. (2.19)). We will also examine the assumption that the ratio of income equals the ratio of IALS scores and also that this ratio is directly w. The relative IALS scores (for reading comprehension – *Mean Prose Literacy*; according to [80]), for selected 18 OECD countries, for which the results are available, are gathered in Tab. 2.7 in subsection 2.5.5.

2.5.3 The sensitivity analysis of the wage rate per unit of human capital

We will now analyze the case of Poland, taking the parameter w – the wage rate per unit of human capital – from 65% to 100%. We take the empirical value of the average length of schooling in Poland $s = 10.75$[1] and the demographic data (zero population growth rate, hence $f = 0$, average length of life $T = 74$). As a working assumption, we take the relative price of capital equal to 1 (the same price of capital as in USA).

Tab. 2.2 gathers the results from the model for human capital at age 6, $6+s$ and the average human capital p.c. of an average Pole, depending on the value of the parameter w. We also give the results for USA for comparison ($w = 1$).

The relative price of physical capital, which is the measure of its quality, influences the results in a significant way. If the wage rates per unit of human capital w were equal for Poland and USA, human capital resources would be similar for both countries, with a slight advantage of Poland. At low values of w, the differences in human capital (for people of the same age) are substantial. The elasticity of human capital with respect to w equals 4.29, which means that a one-percent increase in w causes a 4.29-percent

Table 2.2: **The dependence of human capital at age 6, 6+s and the average human capital p.c. for Poland on the wage rate per unit of human capital**

Country	w	$h(6)$	$h(6+s)$	$\bar{h}$
USA	1.000	4657.98	25115.75	40304.40
	0.650	936.47	4077.53	6700.03
	0.678	1122.00	4885.35	8027.41
	0.700	1286.55	5601.85	9205.42
	0.750	1729.19	7529.13	12371.58
	0.800	2280.16	9928.15	16313.54
Poland	0.850	2956.67	12873.79	21153.71
	0.900	377.37	16447.25	27025.48
	0.950	4762.37	20736.07	34072.70
	0.969	5184.19	22572.75	37090.65
	1.000	5933.26	25834.31	42453.05

[1]We obtained this value in the following way. In 2002 the population structure with respect to education was (according to the main statistical office of Poland GUS): higher education 10.2%; secondary 32.6%; vocational 24.1%; primary 28.2%; incomplete primary 4.9%. These types of education were attributed, respectively, 17, 12, 11, 8 and 4 years of schooling. Then, a weighted average was calculated. An analogous procedure for 1988 gives $s = 10.03$ years.

increase in human capital:

$$\bar{h} = 42451 w^{4.2857}. \tag{2.25}$$

The elasticity of GDP p.c. with respect to w is even higher (see Tab. 2.3) and equals 5.29:

$$y = 38111 w^{5.2857}. \tag{2.26}$$

If the wage rates per unit of human capital were equal in Poland and USA, Poland would have higher GDP p.c., resulting from higher average human capital p.c. This difference would not be very big, however. Low values of w, at 65-70%, would lead to the Polish GDP p.c. at 10-20% of the American GDP p.c.

2

The dependence of TFP and the physical capital-human capital ratio on the wage rate per unit of human capital is linear (Tab. 2.3). At low values of w, to account for the lower value of GDP p.c. we need the assumption of 20% lower TFP in Poland than in USA. However, if we take a higher relative price of capital in Poland, we need a much smaller difference in technology.

Table 2.3: **The dependence of GDP p.c., TFP and the physical capital-human capital ratio κ in Poland on the wage rate per unit of human capital. GDP p.c. and TFP in relative values (1 for USA)**

Country	w	GDP p.c.	TFP	κ
USA	1.000	1.000	1.000	3.537
Poland	0.650	0.108	0.744	2.299
	0.678	0.135	0.766	2.398
	0.700	0.160	0.783	2.476
	0.750	0.230	0.821	2.653
	0.800	0.324	0.858	2.830
	0.850	0.446	0.894	3.007
	0.900	0.604	0.930	3.184
	0.950	0.803	0.965	3.360
	0.969	0.892	0.979	3.428
	1.000	1.053	1.000	3.537

2.5.4 Conclusions for Poland with *empirical* and *calibrated* wage rates per unit of human capital

We will now present the conclusions from the model for human capital in Poland, taking two different estimates of the wage rate per unit of human capital. Firstly, we will use the *empirical* value $w = 0.969$, obtained from Hendricks' data [38], who claims that the average wage of a Polish immigrant on the American labour market equals 92.3% of the American's wage, provided that the American has the same nominal education (length of schooling). Secondly, we will calibrate the wage rate per unit of human capital in such a way that the model gives the empirical value of Polish GDP p.c. (13.5% of the American value, as of 2000). According to Tab. 2.3, the *calibrated* value of w equals 0.678.

Figures 2.5 – 2.7 show the growth paths of human capital and resources allocated to human capital accumulation for the representative individual in Poland (for $w = 0.678$ and $w = 0.969$) and the comparison with the United States ($w = 1$, by definition).

The average human capital of a Pole of any age (or the average over all ages) equals around one fifth of their American peer's value, if we take the *calibrated* value of w and around 90% for the *empirical* w. Thus, if we

Figure 2.5: **The comparison of the growth paths of human capital for the representative individuals in Poland and USA**

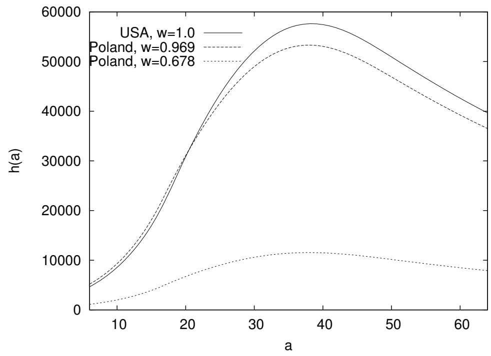


Figure 2.6: **The comparison of the growth paths of the fraction of time allocated to human capital accumulation for the representative individuals in Poland and USA**

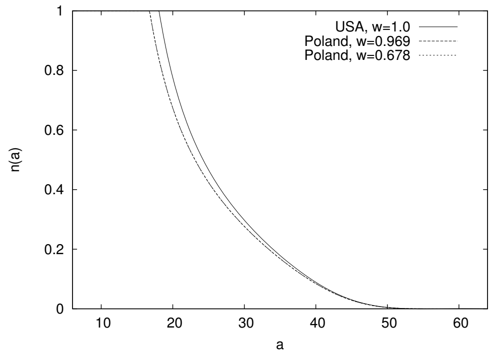
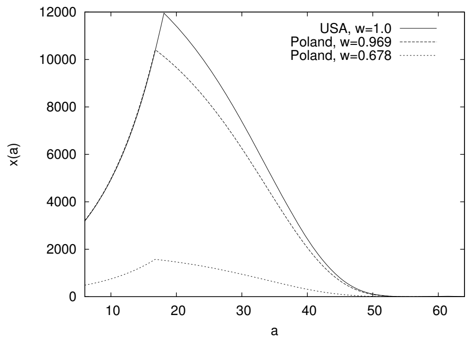


Figure 2.7: **The comparison of the growth paths of the expenditures on market goods allocated to human capital accumulation for the representative individuals in Poland and USA**



take $w = 0.678$, we are forced to conclude that the quality of human capital for Poles is much inferior to the Americans' human capital quality. In other words, one year in the Polish school endows pupils with much smaller amount of human capital than one year in an American school. If $w = 0.969$, then we have comparable qualities of human capital in Poland and in USA.

The American, who finishes their education a little later, allocates more of their time to human capital formation than their Polish peer, thus devotes a little less time for work. These differences are, however, not very large and almost meaningless from the point of view of human capital.

The differences in expenditures on human capital accumulation are substantial. For $w = 67.8\%$, the expenditures of a Pole are on average six times smaller than of an American. This results from the much lower quality of human capital of a Pole, who earns much less and thus has less to spend on human capital accumulation. If, however, $w = 96.9\%$, then the level of expenditures is only a little lower in the case of the representative Pole.

To compare the TFP and the physical capital-human capital ratio κ in Poland and in USA, we have to fix the relative price of physical capital in eq. (2.20). The product $p_k(r + \delta_k)$ equals the marginal product of physical capital (the right-hand side of (2.20)). In USA $r = 7\%$, $\delta_k = 6\%$, so the price of capital ($p_k = 1$) $r + \delta_k = 13\%$. In the case of Poland, we will check the model predictions for three values of p_k:

Table 2.4: **The level of technology A and the physical capital-human capital ratio κ in USA and Poland for different values of the relative price of capital p_k, at $w = 67.8\%$**

Country	p_k	A	κ
USA	1.0	1.000	3.537
Poland	1.0	0.766	2.398
	1.2	0.811	1.999
	2.3	0.996	1.043

Table 2.5: **The level of technology A and the physical capital-human capital ratio κ in USA and Poland for different values of the relative price of capital p_k, at $w = 96.9\%$**

Country	p_k	A	κ
USA	1.0	1.000	3.537
Poland	1.0	0.979	3.428
	1.2	1.036	2.856
	2.3	1.272	1.490

- $p_k = 1$ (no difference in the relative price of physical capital; thus we isolate the influence of human capital),
- $p_k = 1.2$ (assumption that physical capital in Poland is 20% more expensive than in USA seems to be reasonable),
- $p_k = 2.3$ (this corresponds roughly to the actual value of the interest rate in Poland in 2000 $r \approx 23\%$).

The results of this experiment are gathered in Tab. 2.4 (for $w = 0.678$) and 2.5 (for $w = 0.969$).

In the case of the *calibrated* value of w, the assumption of equal prices of capital in Poland and in USA leads to the TFP in Poland at 77% of the American value. This means that the levels of technology in Poland and USA do not differ so much, as in the case of the human capital resource. The physical capital-human capital ratio is lower in Poland. The assumption of 20% more expensive capital in Poland gives similar values of TFP and

2

Table 2.6: **The comparison of empirical data for the countries analyzed by Manuelli and Seshadri, ranked with respect to GDP p.c.**

Decile	GDP p.c.	s	s_{model}	T	$e^f/2$	p_k
90-100	0.921	10.93	11.64	78	0.85	1.02
80-90	0.852	9.94	10.92	76	0.90	1.11
70-80	0.756	9.72	10.40	73	1.00	1.06
60-70	0.660	8.70	9.64	71	1.20	1.04
50-60	0.537	8.12	8.90	69	1.35	1.52
40-50	0.437	7.54	6.79	64	1.60	1.77
30-40	0.354	5.88	5.69	57	2.05	1.56
20-30	0.244	5.18	4.29	54	2.50	1.93
10-20	0.146	4.64	3.01	51	2.70	2.11
0-10	0.052	2.45	2.19	46	3.10	2.78

Explanation: the table includes the average length of schooling (s), average length of schooling that results from the model s_{model} (at calibrated values of GDP p.c.), the average length of life (T), the average number of children per person ($e^f/2$) and the relative price of physical capital with repsect to USA (p_k).
Source: Manuelli and Seshadri [59].

κ. Taking the actual interest rate in Poland ca. 2000 means that we do not need a difference in TFP to account for the difference in GDP p.c. – the combined effects of human and physical capital are enough. For this case, the physical capital and human capital are similar in Poland and in USA.

In the case of the *empirical* value of w, the assumption of equal prices of capital in Poland and USA leads to similar values of TFP and κ. At 20% more expensive physical capital in Poland, the equilibrium conditions imply a higher value of TFP in Poland and a lower physical capital-human capital ratio. If we take the relative price of capital that corresponds to the actual interest rate in Poland ca. 2000, then the Polish TFP has to be 27% higher than in USA, which is not plausible, since the United States are commonly believed to be the technological leader.

The situation of Poland can be assessed also with respect to other countries analyzed by Manuelli and Seshadri (Tab. 2.6). The authors of the model divided the countries according to GDP p.c. and calculated the average length of schooling, the relative price of capital and demographic parameters. Poland can be situated in the penultimate decile with respect to GDP p.c. The average length of schooling for the countries with a similar GDP p.c. is 4.64 (10.75 for Poland), the average lifespan – 51 years (74 in Poland), the average number of children per person – 2.7 (less than 1 in the case of Poland), the average relative price of capital 2.11 (this value seems to be appropriate for Poland, if we take the actual interest rate in Poland in 2000). Such parameters (except for the high value of the relative price of capital) situate Poland among countries whose GDP p.c. is around 80-90% of the American value. Such value of GDP p.c. (89.2% of the American value) is also obtained if we take the *empirical* value of $w = 96.9\%$, which results from the fact that in such case the average human capital per person in Poland is around 92% of its value in USA.

Therefore, we encounter a paradox: the low GDP p.c. in Poland suggests poor quality of human capital in Poland, but then, how can we explain the fact that the Polish immigrants get such high salaries, just a little shy of the American natives' (with similar nominal education) ones, if their human capital is of such low quality? Assuming that a Pole gets 96.9%

for one unit of his human capital instead of 67.8% he should get means that we undermine the role of free market as a perfect measure of value. Possibly, some political or social factors make it impossible to offer such low salaries to Polish immigrants, since the employers would be accused of abusing them and treating them as a cheap workforce. In other words, the representative Pole's human capital is of much worse quality than the representative American's with identical nominal length of schooling, but the salaries don't fully capture this difference in quality.

However, the values of w equal to 96.9% and 67.8% give salaries, respectively, at 92.3% (empirical value given by Hendricks) and merely around 14% of the average American's salary. Is the market so unreliable that it values the work of an average Polish immigrant at 92% of the average American's value instead of 14% - almost sevenfold more than it is worth, taking the low quality of the Polish immigrant's human capital into consideration? It seems impossible that the market inefficiencies could be so high.

Presumably, the value of w which mirrors the actual quality of human capital in Poland is closer to the *empirical* value of 96.9%, not the shockingly low 67.8%. Assuming this, we get the stock of human capital for Poland just shy of the American and almost equal values of Polish and American GDP p.c. The model is thus unable to predict the actual value of Polish GDP p.c. correctly. The reasons for the divergence of the model value (close to the American value) and the actual value (around 13.5% of the value for USA) have to be looked for outside the model.

The model fails to take into consideration the fact that the market economy in Poland is relatively young – only around twenty years have passed since its launch, which appears to be fairly important. Firstly, the time span might have been too short for equilibrium to be achieved. In such case, equations (2.20)-(2.22) should be modified to allow for this effect. Secondly, the structure of Polish economy is not yet human capital-oriented, i.e. relatively high stock of human capital in Poland is not fully used in production. Thirdly, the Manuelli-Seshadri model does not take into account the differences in the levels of technology – the TFP estimates result only from equilibrium considerations for the physical capital and labour market.

In the next subsection we will continue the analysis, taking the differences in technology levels into account.

2.5.5 The analysis for the OECD countries based on the estimates of the wage rate per unit of human capital from the immigrants' earnings and the IALS scores

In this subsection we will perform the analysis for 29 OECD countries[2]. For 19 countries the results of the IALS scores are available (hence, we will term them the *IALS countries*), and for 25 countries we have the data about immigrants' earnings.

Tab. 2.7 gathers the results of four procedures to estimate the wage rate per unit of human capital.

- w_1 – calculated from the immigrants' earnings on the American labour market. The procedure to estimate w was given in subsection 2.5.2.
- w_2 – taken as the relative IALS score (the IALS score for USA = 1).

[2] At the beginning of the year 2000 OECD had 29 members. At the end of 2000, Slovakia joined, but it is not included in this research.

- w_3 – calculated from the assumption that the relative wage is equal to the relative IALS score. The procedure to estimate w is similar to the case of w_1.

- w_4 – according to Coulombe *et al.* [26], a 1-percent decrease in the IALS score corresponds to a 2.5-percent loss of productivity. Hence, we assume here that the ratio of the wages (given country/USA) equals the ratio of productivities, as given by the IALS test. For example, for Poland the relative IALS score equal to 0.807 corresponds to the relative productivity of 0.5175. Taking such ratio of wages gives $w = 0.858$. This method of estimating w seems to be theoretically soundest.

Table 2.7: **The estimate of the wage rate per unit of human capital for 18 IALS countries (for the 19th IALS country – USA – $w = 1$ by definition)**

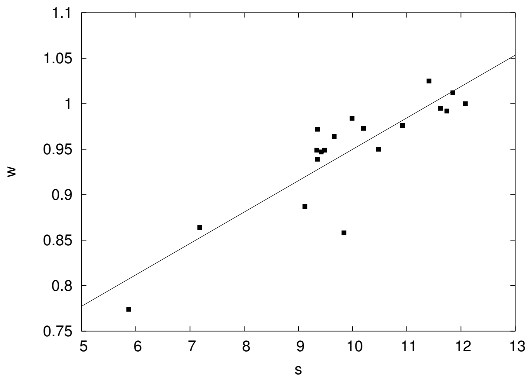
Country	s	immigrants' wage in USA (USA = 1)	w_1	IALS score w_2	w_3	w_4
POR	5.87	1.094	0.940	0.757	0.876	0.774
ITA	7.18	1.191	0.971	0.856	0.912	0.864
HUN	9.12	1.004	0.964	0.859	0.936	0.887
BEL	9.34	1.265	1.010	0.964	0.960	0.949
IRL	9.35	1.193	0.999	0.944	0.956	0.939
NED	9.35	1.102	0.984	1.012	0.968	0.972
UK	9.42	1.305	1.017	0.958	0.959	0.947
CZE	9.48	1.005	0.968	0.962	0.960	0.949
DEN	9.66	1.314	1.021	0.988	0.968	0.964
POL	9.84	0.923	0.957	0.807	0.933	0.858
FIN	9.99	–	–	1.024	0.978	0.984
GER	10.20	1.170	1.006	0.992	0.975	0.973
SWI	10.48	1.314	1.032	0.939	0.968	0.950
AUS	10.92	1.313	1.037	0.982	0.981	0.976
SWE	11.41	1.292	1.041	1.078	1.006	1.025
CAN	11.62	1.258	1.039	1.001	0.995	0.995
NZL	11.74	1.262	1.040	0.993	0.994	0.992
NOR	11.85	1.310	1.049	1.033	1.003	1.012

Explanation: w_1 – w estimated from the immigrants' earnings, w_2 – from the IALS score, w_3 – the IALS score = the relative immigrants' earnings, w_4 – the relative productivity from the IALS score. The countries are ranked with respect to the average length of schooling s. The country acronyms are explained in Appendix B. The value of s for Poland differs from the one in subsection 2.5.3 due to a different methodology in Barro and Lee [10]. The IALS data from 1998. The data about the average years of schooling and the immigrants' earnings from 2000.
Source: Barro and Lee [10], Hendricks [38], Coulombe *et al.* [26], own calculations.

Taking the sensitivity of the model predictions with respect to the wage rate per unit of human capital, the discrepancies in the values of w are rather large.

For the countries for which the IALS test had not been performed (*non-IALS countries*), the wage rate per unit of human capital was estimated by two methods. The values of w_1 were found in the standard way, from the immigrants' earnings on the American labour market. The value of w_4

Figure 2.8: **The dependence of the wage rate per unit of human capital and the average years of schooling for 18 OECD countries**



was found from the regression equation for the wage rate per unit of human capital vs. the average length of schooling s for 19 IALS countries (the dependence of w vs. s is given in Fig. 2.8):

$$w(s) = \underset{(0.047770)}{0.604857} + \underset{(0.004749)}{0.034503}s \tag{2.27}$$

The regression coefficients in eq. (2.27) are statistically significant. The values of w_4 from eq. (2.27), as well as the values of w_1 are gathered in Tab. 2.8. Also for these countries, the discrepancies in the estimates of w are large.

Table 2.8: **The estimate of the wage rate per unit of human capital for 10 non-IALS OECD countries**

Country	s	immigrants' wage in USA (USA = 1)	Estimate w_1	Estimate w_4
TUR	5.29	1.070	0.929	0.787
MEX	7.23	0.765	0.894	0.854
SPA	7.28	1.055	0.950	0.856
FRA	7.86	1.265	0.991	0.876
AUT	8.35	1.263	0.997	0.893
LUX	8.49	—	—	0.898
GRE	8.67	1.026	0.962	0.904
ISL	8.83	—	—	0.910
JPN	9.47	1.364	1.026	0.932
KOR	10.84	0.776	0.939	0.979

Explanation: w_1 – from the immigrants' wages, w_4 – from the regression equation (2.27). The countries are ordered with respect to the average length of schooling s. The value of s for Luxembourg is not known – we take the value for the closest country in terms of education structure, which is Argentina. The data on s and the immigrants' earnings from 2000.
Source: Barro and Lee [10], Hendricks [38], own computations.

In Tabs. 2.9 and 2.10, we have gathered the conclusions from the Manuelli-Seshadri model about human capital and GDP p.c.[3] for 18 *IALS countries*. The measure of the quality of estimate is the mean absolute error (MAE), calculated as:

$$MAE = \frac{1}{N}\sum_{i=1}^{N}|y_i^{est} - 1|, \tag{2.28}$$

where N is the number of countries and y_i^{est} the estimated value of GDP p.c. for the i-th country (the value of 1 corresponds to an exact agreement with the empirical value of GDP p.c.).

Table 2.9: **The conclusions from the Manuelli-Seshadri model for the average human capital per person h (with respect to h in USA) for 18 IALS countries**

Country	s	h at:			
		w_1	w_2	w_3	w_4
		(with respect to USA)			
POR	5.87	0.953	0.379	0.704	0.414
ITA	7.18	1.052	0.613	0.804	0.638
HUN	9.12	0.955	0.582	0.841	0.668
BEL	9.34	1.153	0.944	0.927	0.883
IRL	9.35	1.097	0.861	0.909	0.842
NED	9.35	1.029	1.160	0.959	0.976
UK	9.42	1.188	0.920	0.924	0.875
CZE	9.48	0.959	0.933	0.925	0.881
DEN	9.66	1.199	1.041	0.954	0.937
POL	9.84	0.902	0.434	0.809	0.565
FIN	9.99	–	1.196	0.982	1.008
GER	10.20	1.100	1.036	0.962	0.953
SWI	10.48	1.214	0.810	0.923	0.851
AUS	10.92	1.210	0.958	0.954	0.933
SWE	11.41	1.214	1.410	1.049	1.136
CAN	11.62	1.194	1.018	1.027	1.027
NZL	11.74	1.197	0.981	0.986	0.977
NOR	11.85	1.236	1.158	1.020	1.060

Source: own calculations based on the data from Tab. 2.7.

As expected, the assessments of the wage rate per unit of human capital based on immigrants' earnings lead to overestimated values of GDP p.c., especially for the poorest countries, including Poland, where we have a three-fold overestimation. This means that the labour market in the United States can not correctly evaluate the immigrants' abilities (and hence their productivity) – the differences in wages are just of the order of 5-15%, which leads to very close estimates of human capital resources. To explain the low actual values of GDP p.c. in many countries, we have to refer to the socio-political factors, as analyzed in subsection 2.5.4.

Using the results of literacy tests, such as IALS, leads to much more realistic assessments of w. The theoretically most justified method, using the empirical relation between the IALS score and productivity, leads to

[3]The statistical data on GDP p.c. [57] take the purchasing power into account (*Purchasing Power Parity*). The value of GDP p.c. for Poland thus differs from the one given in subsection 2.5.3.

GDPs p.c. that are actually rather close to the empirical values. Hence, also the estimates of human capital resources are the most realistic.

Table 2.10: **The conclusions from the Manuelli-Seshadri model for GDP p.c. (with respect to GDP p.c. in USA) for 18 IALS countries**

Country	s	GDP p.c. empirical (USA = 1)	GDP p.c. at:			
			w_1	w_2	w_3	w_4
			(with respect to the empirical GDP)			
POR	5.87	0.471	1.904	0.610	1.311	0.682
ITA	7.18	0.673	1.517	0.780	1.089	0.819
HUN	9.12	0.261	3.529	1.918	3.019	2.275
BEL	9.34	0.765	1.522	1.190	1.163	1.095
IRL	9.35	0.808	1.357	1.007	1.076	0.978
NED	9.35	0.743	1.362	1.580	1.249	1.278
UK	9.42	0.654	1.847	1.347	1.355	1.268
CZE	9.48	0.456	2.034	1.968	1.946	1.832
DEN	9.66	0.758	1.615	1.357	1.218	1.191
POL	9.84	0.225	3.831	1.558	3.351	2.153
FIN	9.99	0.704	–	1.739	1.364	1.409
GER	10.20	0.714	1.551	1.441	1.315	1.301
SWI	10.48	0.862	1.453	0.881	1.035	0.938
AUS	10.92	0.695	1.807	1.355	1.348	1.312
SWE	11.41	0.674	1.874	2.254	1.565	1.728
CAN	11.62	0.751	1.653	1.357	1.372	1.372
NZL	11.74	0.578	2.151	1.686	1.694	1.675
NOR	11.85	1.081	1.200	1.106	0.946	0.993
mean absolute error			91.7%	49.7%	54.0%	43.2%

Source: own calculations based on the data from Tab. 2.7, Malaga [57].

Table 2.11: **The conclusions from the Manuelli-Seshadri model for the average human capital per person h (with respect to h in USA) for 10 non-IALS countries**

C'try	s	GDP p.c. empirical (USA = 1)	h at:		GDP p.c. at:	
			w_1	w_4	w_1	w_4
			(w.respect to USA)		(w.r. to GDP$_{emp}$)	
TUR	5.29	0.200	0.922	0.445	4.283	1.749
MEX	7.23	0.277	0.736	0.605	2.380	1.870
SPA	7.28	0.533	0.955	0.611	1.700	0.980
FRA	7.86	0.690	1.122	0.661	1.613	0.840
AUT	8.35	0.722	1.133	0.707	1.565	0.874
LUX	8.49	1.225	—	0.721	—	0.528
GRE	8.67	0.457	0.962	0.737	2.023	1.457
ISL	8.83	0.847	—	0.753	—	0.809
JPN	9.47	0.731	1.226	0.812	1.719	1.035
KOR	10.84	0.455	0.801	0.958	1.655	2.065
mean absolute error					109.4%	41.4%

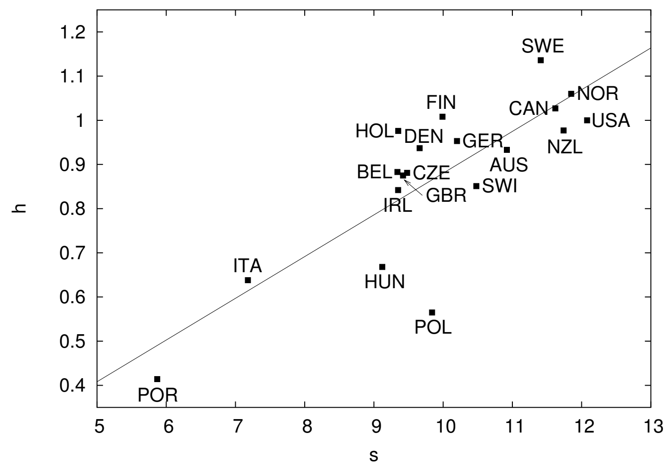
Source: own calculations based on the data from Tab. 2.8, Malaga [57].

The average human capital resource in the *IALS countries* are close to one another and differ by 5-15%. However, we observe much lower resources in Italy, Portugal, Hungary and Poland. In the first two countries it results from a relatively short average length of schooling (6-7 years). In Hungary and especially in Poland, the lower human capital resources result more from bad quality of schooling – the average length of schooling is close to the European average (9-10 years).

In the *non-IALS countries* (Tab. 2.11), a low level of human capital is observed for countries with short average length of schooling – Turkey, Mexico and Spain. However, we can not thus conclude about the quality of schooling in these countries, since we have implicitly assumed in the w estimates that the quality is similar (we have assumed linear dependence of w on s (eq. (2.27))).

For the *IALS countries*, we also analyzed the relationship between the average length of schooling and the average human capital resource per person, shown in Fig. 2.9. The straight line was obtained from linear regression and expresses the average human capital level resulting from a given period

Figure 2.9: **The average length of schooling and the human capital resource p.c. for 18 *IALS countries***



of schooling, assuming similar qualities of schooling. The countries that lie above the regression line can be interpreted as the countries with higher than average quality of schooling, i.e. to reach a specified human capital resource, one needs a shorter period of schooling. Thus, we conclude that The Netherlands, Denmark, Finland and Sweden have the best quality of schooling. The countries that are located below the regression curve are the ones where the quality of schooling is lower than average. The relatively lowest quality of schooling is observed for Poland (which lies the furthest away from the regression line) – hence the level of human capital of an average Pole is much lower than of a representative person from other countries with similar average length of schooling. The quality of schooling is rather low also in Hungary, USA, Switzerland, Portugal and New Zealand.

In Tabs. 2.12 and 2.13 we gather the conclusions from the Manuelli-Seshadri model on TFP. Independently of the method of estimating the wage rate per unit of human capital, the observed differences in TFP do not exceed 20% and are typically just a few percent for most of the countries. This results from the fact that technology is not explicitly modelled in the

Table 2.12: **The conclusions from the Manuelli-Seshadri model – TFP for different values of w for 18 IALS countries**

Country	s	TFP at: w_1	w_2	w_3	w_4
		(USA = 1)			
POR	5.87	0.958	0.827	0.913	0.839
ITA	7.18	0.980	0.899	0.938	0.904
HUN	9.12	0.975	0.901	0.955	0.921
BEL	9.34	1.006	0.975	0.972	0.964
IRL	9.35	0.999	0.961	0.969	0.957
NED	9.35	0.989	1.008	0.971	0.980
UK	9.42	1.011	0.971	0.971	0.963
CZE	9.48	0.978	0.973	0.972	0.964
DEN	9.66	1.014	0.991	0.978	0.975
POL	9.84	0.970	0.863	0.953	0.900
FIN	9.99	–	1.016	0.984	0.989
GER	10.20	1.004	0.994	0.982	0.981
SWI	10.48	1.021	0.957	0.978	0.965
AUS	10.92	1.025	0.987	0.987	0.983
SWE	11.41	1.027	1.052	1.004	1.017
CAN	11.62	1.026	1.000	1.002	1.002
NZL	11.74	1.027	0.995	0.995	0.994
NOR	11.85	1.033	1.022	1.002	1.008

Source: own calculations based on the data from Tab. 2.7.

Manuelli-Seshadri model, but only stems from the equilibrium equations for the physical and human capital markets. It seems that we encounter here a contradiction with the empirical data, which suggest much larger discrepancies in technology levels. Thus, we should not conclude here about the technological advancement of the respective countries.

Summing up, we should emphasize that the mean absolute errors of our estimations in this chapter are rather high, especially for the poorer countries, particularly the ones which underwent political transformation in the recent years. It seems thus that the Manuelli-Seshadri model in its original form can not describe the whole economy in a satisfying way, since it oversimplifies the problem of differences in the levels of technology between the countries. The differences in the levels of human capital are not enough to account for the differences in the GDP p.c. levels. Human capital is, however, for sure a very important factor of growth and the presented way of modelling, which takes the quantitative and qualitative aspects into account, is the most advanced way to model human capital. The analysis of human capital should be backed up with the analysis of the influence of technological factors and their role in economic growth. Therefore, the next chapter is devoted to the Manuelli-Seshadri model with technological progress (and other models of technological progress).

2

Table 2.13: **The conclusions from the Manuelli-Seshadri model – TFP for different values of w for 10 non-IALS countries**

Country	s	TFP at:	
		w_1	w_4
		(USA = 1)	
TUR	5.29	0.950	0.848
MEX	7.23	0.926	0.897
SPA	7.28	0.965	0.899
FRA	7.86	0.993	0.913
AUT	8.35	0.998	0.925
LUX	8.49	—	0.929
GRE	8.67	0.973	0.933
ISL	8.83	—	0.937
JPN	9.47	1.017	0.952
KOR	10.84	0.957	0.985

Source: own calculations based on the data from Tab. 2.8.

2.6 Summary

In this chapter, we have analyzed the human capital model of Manuelli and Seshadri and we performed an empirical analysis of the role of human capital in contemporary economies.

We showed the role of the most important parameter of the model – the wage rate per unit of human capital – and we presented the methods to estimate it, basing on the immigrants' earnings in USA and the results of adult literacy tests.

The principal theoretical progress that can be attributed to the Manuelli-Seshadri approach consists in introduction and quantification of the notion of the quality of human capital. It was shown that there are countries with a much better or much worse quality of human capital. The former includes e.g. Sweden, Finland and The Netherlands and the latter e.g. Poland.

We also found that the human capital resource of an average Pole is only a little lower than of an average American – thus this difference can not explain the large difference in GDP p.c. between the two countries. The analysis for other OECD countries supports the conclusion that human capital is a very important growth factor, but to describe economic growth in a satisfying manner one also has to supplement the analysis with such factors as technological progress, which is the task for the next chapter.

Chapter 3

Technological progress – the standard approach

This chapter is devoted to the role of technological progress for economic growth. We will examine the implications of the Manuelli-Seshadri model with a modified production function. Then, we will introduce a technological progress model with technology diffusion. The last section will concern an endogenous model of technological progress. All of these models will be confronted with empirical data for the group of 29 OECD countries.

3.1 The modified production function

In the previous chapter, we analyzed the Manuelli-Seshadri model with the following production function:

$$F(k,h) = Ak^{\theta}h^{1-\theta}, \tag{3.1}$$

in which the differences in the technology index A resulted from the equilibrium conditions for the physical and human capital markets, i.e. the value of A was adjusted in such a way that on both of these markets we had the equality of the marginal costs and products. This led to small differences in the values of A (of the order of a few percent) and thus to the conclusion that the role of the differences in technology was small. In this way, however, we obtained paradoxical conclusions for the less-developed countries (like e.g. Poland) – to account for the differences in GDP p.c. we needed *dramatically* low human capital resources, which is not confirmed by such empirical data as the immigrants' earnings in USA or adult literacy tests (IALS). Therefore, it seems that the differences in GDP p.c. can not be fully explained by the differences in human capital. The assumption that one also needs some differences in technology levels is intuitive – the United States equipped with e.g. the Polish technology of production would certainly not reach such high GDP p.c. Moreover, the differences in technology are confirmed by most of empirical research.

In this section, we will first use the Romer model to introduce a modified production function that takes such differences into account. Then, we will analyze the Manuelli-Seshadri model with such production function.

3.1.1 Setup of the model

Let us assume that we have two factors of production – physical capital k and human capital h. The level of technology – the range of intermediate goods that are usable in the economy – is given by the variable A. We assume that one unit of an intermediate good can be exchanged for one unit of physical capital (i.e. $k = Ax$). The Romer production function (1.93) thus takes the form:

$$F(A,k,h) = k^{\theta}\left(\nu hA\right)^{1-\theta}, \tag{3.2}$$

where ν is the fraction of human capital that is used in production (the remaining part, $1-\nu$, is used in the production of technology). The knowledge accumulation equation takes the form:

$$\dot{A} = \zeta(1-\nu)hA, \tag{3.3}$$

where ζ denotes a parameter which expresses the maximal growth rate of technology (the innovation rate).

Equation (1.101) implies that the socially optimal stationary growth path satisfies:

$$h_A = (1-\nu)h = \frac{\zeta h - r}{\zeta \epsilon}, \tag{3.4}$$

where h_A is the human capital resource used in the technological sector. Hence, from eq. (1.102) we can find the simultaneous growth rate of technology, physical capital, consumption and production p.c.:

$$g = \zeta h_A = \frac{\zeta h - r}{\epsilon}. \tag{3.5}$$

Knowing the rate of technology growth, we can find the innovation rate ζ:

$$\zeta = \frac{g\epsilon + r}{h}. \tag{3.6}$$

Let us now set $\epsilon = 1$ (it corresponds to the logarithmic utility of consumption function[1]). The fraction of human capital used in the production sector equals:

$$\nu = 1 - \frac{\zeta h - r}{\zeta h} = \frac{r}{\zeta h}. \tag{3.7}$$

Let us assume that the above equations describe a certain country which is a reference country for the analysis (this is USA in the empirical analysis). Thus, for an arbitrary country we have:

$$\nu' = \frac{r}{\zeta' h'} = \nu \frac{\zeta}{\zeta'} \frac{h}{h'}, \tag{3.8}$$

where the primed quantities refer to this country. Let us define the quantity:

$$\psi \equiv \frac{g' + r}{g + r}, \tag{3.9}$$

which expresses the influence of the technological progress on the innovation rate:

$$\frac{\zeta'}{\zeta} = \psi \frac{h}{h'}. \tag{3.10}$$

If the rate of technological progress in the country under investigation is higher than in USA, we have $\psi > 1$. The other factor, h/h', expresses the influence of human capital on the innovation rate.

Substituting (3.10) into (3.8), we get:

$$\nu' = \frac{\nu}{\psi}. \tag{3.11}$$

For a country with $\psi < 1$, a relatively larger fraction of human capital is thus employed in the production sector. A relatively smaller part is used in the research sector. If $h/h' > 1/\psi$, the efficiency of the research sector

[1] $\lim_{\epsilon \to 1} \frac{C^{1-\epsilon}-1}{1-\epsilon} = \ln C$.

is, however, higher, i.e. the parameter ζ' is higher. In further course of this section we will consider the source of the larger efficiency of the research sector in some of the OECD countries with respect to USA. This fact seems to be in contradiction with the empirical data on expenditures on research and development.

Taking (3.11) into account, the production function for the analyzed country takes the form:

$$F(A,k,h) = k^{\theta}\left(\nu' hA\right)^{1-\theta} = k^{\theta}\nu^{1-\theta}\left(hA\right)^{1-\theta}\psi^{\theta-1}, \tag{3.12}$$

where the factor $\nu^{1-\theta}$ is constant for all countries and in this way can be dropped out (we are interested only in relative differences in GDP p.c., related to different values of k, h, A and ψ in various countries). Finally, we obtain:

$$F(A,k,h) = k^{\theta}\left(hA\right)^{1-\theta}\psi^{\theta-1}. \tag{3.13}$$

3

3.1.2 The empirical analysis for the OECD countries

In Tabs. 3.1 and 3.2 we gather the data on the levels of technology and the growth rates of technology, as well as the resulting values of the parameters ψ and the ratios of the innovation rates ζ'/ζ for, respectively, 19 *IALS countries* and 10 *non-IALS countries*. The method of estimating A and g' for the analyzed countries is described in Appendix B.

Equation (3.10) implies that the ratio of the innovation rates ζ'/ζ is deaggregated to the *technological progress effect* ψ and the *human capital effect* h/h' (which expresses how many times the American human capital resource (h) is larger than in the analyzed country (h')). For almost all countries $\zeta' > \zeta$. In the conventional approach to technology, where its growth rate depends on R&D expenditures, the relation $\zeta' > \zeta$ seems to be paradoxical – the American R&D expenditures are one of the highest in the world. However, this approach does not take the effects of technology diffusion into account. High R&D expenditures are not necessary to have a high growth rate, provided that one can imitate technological solutions and innovations from other countries. For example, a computer had to be invented only once, in USA, and other countries soon introduced this technological solution imitating the American invention.

The higher values of ζ' in the analyzed countries than in USA should thus be interpreted as the joint effect of R&D expenditures (usually lower than in USA) and technology diffusion, i.e. technology transfers from technology leading countries. The countries that benefit the most from technology diffusion are, according to the model: Portugal, Ireland and Turkey ($\zeta'/\zeta > 2$). A very fast technology growth is observed also in Italy, the Czech Republic, Poland, Mexico, Spain, France, Luxembourg, Greece, Iceland and South Korea ($\zeta'/\zeta > 1.5$). In Sweden, Canada, New Zealand and USA the innovation rate is the lowest ($\zeta' \approx \zeta$). Thus, one can conclude that the latter are rather the suppliers of technology. Some detailed conclusions about the effects of R&D expenditures and technology diffusion can, however, be drawn only after we explicitly model the research activity and technology diffusion (the imitation activity), which we will examine in next sections.

3

Table 3.1: **The technology level A, technological progress rate g' and the resulting values of ψ for 19 IALS countries in 1999**

Country	s	GDP p.c. (USA = 1)	A	g' (%)	ψ	h/h'	ζ'/ζ
POR	5.87	0.471	0.410	3.51	1.267	2.413	3.058
ITA	7.18	0.673	0.699	1.69	1.048	1.568	1.643
HUN	9.12	0.261	0.257	-0.87	0.739	1.497	1.106
BEL	9.34	0.765	0.808	2.71	1.031	1.133	1.327
IRL	9.35	0.808	0.820	7.02	1.691	1.188	2.009
NED	9.35	0.743	0.748	2.97	1.202	1.025	1.232
UK	9.42	0.654	0.708	1.93	1.077	1.143	1.231
CZE	9.48	0.456	0.583	5.38	1.493	1.136	1.695
DEN	9.66	0.758	0.809	2.40	1.133	1.067	1.209
POL	9.84	0.225	0.239	1.58	1.034	1.770	1.831
FIN	9.99	0.704	0.790	4.26	1.358	0.992	1.347
GER	10.20	0.714	0.694	1.79	1.060	1.049	1.112
SWI	10.48	0.862	0.836	2.27	1.117	1.175	1.312
AUS	10.92	0.695	0.656	2.05	1.092	1.072	1.170
SWE	11.41	0.674	0.761	2.32	1.124	0.880	0.990
CAN	11.62	0.751	0.738	1.17	0.985	0.974	0.959
NZL	11.74	0.578	0.565	1.16	0.983	1.023	1.006
NOR	11.85	1.081	0.793	3.29	1.241	0.943	1.171
USA	12.08	1.000	1.000	1.29	1.000	1.000	1.000

Explanation: GDP p.c. and A with respect to USA, g', h' (for w_4) and ζ' refer to a given country, h and ζ to USA. The method of estimating the technology parameters is given in Appendix B.
Source: Barro and Lee [10], Hendricks [38], own calculations based on Malaga [57].

Table 3.2: **The technology level A, technological progress rate g' and the resulting values of ψ for 10 non-IALS countries in 1999**

Country	s	GDP p.c. (USA = 1)	A	g' (%)	ψ	h/h'	ζ'/ζ
TUR	5.29	0.200	0.186	1.87	1.070	2.247	2.403
MEX	7.23	0.277	0.266	2.38	1.131	1.653	1.869
SPA	7.28	0.533	0.512	2.67	1.166	1.637	1.909
FRA	7.86	0.690	0.712	1.44	1.018	1.512	1.539
AUT	8.35	0.722	0.659	1.55	1.031	1.415	1.460
LUX	8.49	1.225	1.118	3.56	1.273	1.387	1.766
GRE	8.67	0.457	0.427	2.70	1.170	1.357	1.588
ISL	8.83	0.847	0.856	2.67	1.166	1.328	1.548
JPN	9.47	0.731	0.646	2.14	1.102	1.232	1.357
KOR	10.84	0.455	0.411	6.45	1.622	1.044	1.693

Explanation: GDP p.c. and A with respect to USA, g', h' (for w_4) and ζ' refer to a given country, h and ζ to USA. The method of estimating the technology parameters is given in Appendix B.
Source: Barro and Lee [10], Hendricks [38], own calculations based on Malaga [57].

In the above considerations, the influence of the innovation rate on GDP p.c. was only given by the parameter ψ, and not ζ'/ζ – only the parameter ψ is the argument of the production function (3.13). Now, we will analyze the influence of the technological parameters A and ψ on the conclusions from the Manuelli-Seshadri model about GDP p.c., taking the production function of the form:

$$F(A,k,h) = \frac{1}{A_{M-S}} k^{\theta} (hA)^{1-\theta} \psi^{\theta-1}, \tag{3.14}$$

where A_{M-S} is the original TFP from the Manuelli-Seshadri model[2]. Introducing the factor $1/A_{M-S}$, we eliminate these oversimplified differences in TFP levels.

Using the measure (2.28), we will examine the GDP p.c. estimates in two variants:

- variant 1: we take differences in A, $\psi \equiv 1$ (i.e. without the effect of different technology growth rates),
- variant 2: we take differences in A and ψ.

The wage rates per unit of human capital are taken as w_4, since this was the way that minimized the errors coming from the need to estimate w.

3

Table 3.3: **The conclusions from the Manuelli-Seshadri model with the modified production function – GDP p.c. (with respect to the empirical GDP p.c.) for 18 IALS countries**

Country	s	GDP p.c.	A	ψ	GDP p.c. in	
					var1	var2
		(USA = 1)			(w.respect to GDP_{emp})	
POR	5.87	0.471	0.410	1.267	0.442	0.375
ITA	7.18	0.673	0.699	1.048	0.709	0.687
HUN	9.12	0.261	0.257	0.739	0.792	0.974
BEL	9.34	0.765	0.808	1.031	0.982	0.881
IRL	9.35	0.808	0.820	1.691	0.892	0.623
NED	9.35	0.743	0.748	1.202	1.068	0.942
UK	9.42	0.654	0.708	1.077	1.039	0.988
CZE	9.48	0.456	0.583	1.493	1.313	0.998
DEN	9.66	0.758	0.809	1.133	1.057	0.970
POL	9.84	0.225	0.239	1.034	0.896	0.876
FIN	9.99	0.704	0.790	1.358	1.213	0.983
GER	10.20	0.714	0.694	1.060	1.032	0.992
SWI	10.48	0.862	0.836	1.117	0.860	0.797
AUS	10.92	0.695	0.656	1.092	0.999	0.941
SWE	11.41	0.674	0.761	1.124	1.409	1.300
CAN	11.62	0.751	0.738	0.985	1.112	1.124
NZL	11.74	0.578	0.565	0.983	1.140	1.153
NOR	11.85	1.081	0.793	1.241	1.106	0.954
mean absolute error					16.2%	14.4%

Source: own calculations based on Tabs. 2.7, 2.12, 3.1, Malaga [57].

In Tabs. 3.3 and 3.4 we gather conclusions from the Manuelli-Seshadri model with the modified production function, concerning GDP p.c. (with respect to the empirical GDP p.c.) for 29 OECD countries. Taking the technological factors into account leads to much better estimates of GDP p.c. – the mean absolute error decreased to 14-16% (for the *IALS countries*) and

[2]The TFP values are gathered in Tabs. 2.12 and 2.13.

Table 3.4: **The conclusions from the Manuelli-Seshadri model with the modified production function – GDP p.c. (with respect to the empirical GDP p.c.) for 10 non-IALS countries**

Country	s	GDP p.c.	A	ψ	GDP p.c. in	
					var1	var2
		(USA = 1)			(w.respect to GDP_{emp})	
TUR	5.29	0.200	0.186	1.070	0.652	0.622
MEX	7.23	0.277	0.266	1.131	0.841	0.773
SPA	7.28	0.533	0.512	1.166	0.690	0.621
FRA	7.86	0.690	0.712	1.018	0.729	0.720
AUT	8.35	0.722	0.659	1.031	0.710	0.695
LUX	8.49	1.225	1.118	1.273	0.613	0.520
GRE	8.67	0.457	0.427	1.170	0.872	0.783
ISL	8.83	0.847	0.856	1.166	0.776	0.699
JPN	9.47	0.731	0.646	1.102	0.806	0.754
KOR	10.84	0.455	0.411	1.622	1.141	0.819
mean absolute error					24.5%	29.9%

Source: own calculations based on Tabs. 2.8, 2.13, 3.2, Malaga [57].

24-30% (for the *non-IALS countries*). This suggests that the role of the technological factors for economic growth is substantial. For the *IALS countries*, variant 2 (differences in A and ψ) implies a slightly smaller mean absolute error than variant 1, which is in agreement with theoretical predictions associated with the model described in subsection 3.1.1. For the remaining 10 OECD countries, we obtain, however, an increase in the mean absolute error when going from variant 1 to 2. In most cases this results from the fact that taking differences in technology levels leads to an underestimation of GDP p.c. with respect to the empirical GDP p.c. and the introduction of the parameter ψ, the value of which for the analyzed countries exceeds 1, leads to a further decrease in the GDP p.c. estimate. The higher value of the mean absolute error in both variants (with respect to the *IALS countries*) suggests, however, that the conclusions for the 10 *non-IALS countries* are much less reliable, which results from the very process of w estimation. For example, if some of these countries have a higher than average quality of schooling (we can suspect this e.g. of France or Japan), then the estimated value of w_4 for these countries should be higher than the one obtained from the regression equation (2.27),which would lead to higher values of GDP p.c. In such case, the introduction of the variable ψ could improve the quality of estimates.

It seems that the actual diffusion effects are too complex to be captured by a single variable in the production function. For most countries the ratio ζ'/ζ exceeds 1 by quite a large amount, which means that the diffusion effects play an important role in technological progress and thus in economic growth. In the next sections, we will consider some more advanced models of technological progress with technology diffusion and we will analyze their empirical implications.

3.2 Technological progress model with technology diffusion

3.2.1 Technology dynamics equation

Let us consider the following differential equation for the technological variable $A(t)$:

$$\dot{A}(t) = \xi A(t)^{1-\gamma} - \delta A(t), \tag{3.15}$$

where: ξ – a parameter that measures the inputs for technology development, δ – the effective depreciation rate of technology, with technology diffusion effect, γ – a parameter that measures the scale effects.

Eq. (3.15) is a Bernoulli-type equation. To transform it to a linear differential equation, we substitute:

$$y = A^{\gamma}. \tag{3.16}$$

Hence:

$$A = y^{1/\gamma}, \qquad dA = \frac{1}{\gamma} y^{\frac{1}{\gamma}-1} dy. \tag{3.17}$$

Then, eq. (3.15) takes the form:

$$\frac{1}{\gamma} y^{\frac{1}{\gamma}-1} \dot{y} = \xi y^{\frac{1}{\gamma}-1} - \delta y^{\frac{1}{\gamma}}. \tag{3.18}$$

After some simplifications, we obtain:

$$\dot{y}(t) + \gamma\delta y(t) = \gamma\xi, \tag{3.19}$$

with an initial condition:

$$y(0) = A(0)^{\gamma}. \tag{3.20}$$

Eq. (3.19) is a non-homogeneous linear differential equation. Let us denote by $Y(s)$ the Laplace transform $\mathscr{L}\{y(t)\}$. Then:

$$\mathscr{L}\{\dot{y}(t)\} = sY(s) - A(0)^{\gamma}, \tag{3.21}$$

$$\mathscr{L}\{\gamma\xi\} = \frac{\gamma\xi}{s}. \tag{3.22}$$

Eq. (3.19) after Laplace transform takes the form:

$$(s + \gamma\delta)\, Y(s) = \frac{\gamma\xi}{s} + A(0)^{\gamma}. \tag{3.23}$$

Hence:

$$Y(s) = \frac{\frac{\gamma\xi}{s} + A(0)^{\gamma}}{s + \gamma\delta} = \frac{\gamma\xi}{s(s+\gamma\delta)} + \frac{A(0)^{\gamma}}{s+\gamma\delta}. \tag{3.24}$$

We now take the inverse Laplace transform:

$$\mathscr{L}^{-1}\{Y(s)\} = y(t) = -\frac{\xi}{\delta}\left(e^{-\gamma\delta t} - 1\right) + A(0)^{\gamma} e^{-\gamma\delta t}. \tag{3.25}$$

Thus, we have:

$$y(t) = \left(A(0)^{\gamma} - \frac{\xi}{\delta}\right) e^{-\gamma\delta t} + \frac{\xi}{\delta}. \tag{3.26}$$

Returning to the variable $A(t)$, we obtain the growth path of the technological variable:

$$A(t) = \left(\left(A(0)^{\gamma} - \frac{\xi}{\delta}\right) e^{-\gamma\delta t} + \frac{\xi}{\delta}\right)^{1/\gamma}, \tag{3.27}$$

which describes the dependence of the time path of technology on expenditures on R&D and the depreciation and diffusion effects.

3.2.2 The form of the diffusion term

We will now assume that if the level of technology in the analyzed country is lower than in the reference country, then the diffusion effect is proportional to the difference in technology levels. Then, the effective depreciation rate can be written as:

$$\delta = \delta_A + d\frac{A(0) - T(0)}{A(0)}, \tag{3.28}$$

where: δ_A – technology depreciation rate, d – coefficient that characterizes the diffusion effect, $T(0)$ – the initial value of the function $T(t)$, which describes the technology of the technological leader.

If $A(0) < T(0)$, the diffusion effect lowers the effective depreciation rate of technology. If we take the depreciation rate of technology to be zero (and its obsolescence is thus expressed e.g. in the prices of the goods produced with this technology), then the effective depreciation is equivalent to diffusion and the technology dynamics equation can be written as:

$$\dot{A}(t) = \xi A(t)^{1-\gamma} + d\frac{T(0) - A(0)}{A(0)}A(t). \tag{3.29}$$

With such an assumption, a twofold difference in the initial levels of technology of the analyzed and the reference country (i.e. $A(0) = T(0)/2$) means that the growth rate of technology in the analyzed country ($\dot{A}/A$) exceeds the growth rate of technology of the technological leader by the value of d.

The model (3.29), in the case $\gamma = 0$, is the Nelson-Phelps model, given by eq. (1.75)[3]. The generalization to the case $\gamma \in (-\infty, 1]$ makes it possible to take into account the influence of the earlier technology on its development. If $\gamma \in (-\infty, 0)$, then we have increasing returns to scale, i.e. the research activity becomes increasingly effective, which is due to the growing knowledge resource, which can be used in current research. If $\gamma \in (0, 1]$, then this influence is in the reverse direction, which corresponds to the assumption that the knowledge that was the easiest to discover has already been discovered and thus research becomes increasingly difficult. The arbitrary assumption $\gamma = 0$, i.e. the assumption of an ideal equilibrium between the two effects of past knowledge, does not seem plausible without empirical verification. The solution to eq. (3.29) is, according to (3.27), given by:

$$A(t) = \left(\left(A(0)^{\gamma} - \frac{\xi}{d\frac{A(0)-T(0)}{A(0)}}\right)e^{-\gamma d\frac{A(0)-T(0)}{A(0)}t} + \frac{\xi}{d\frac{A(0)-T(0)}{A(0)}}\right)^{1/\gamma}. \tag{3.30}$$

The assumption that the factor $\frac{A(0)-T(0)}{A(0)}$ is fixed throughout the whole period of analysis, which is needed to find the analytical solution to eq. (3.29), suggests the following application of the model. We substitute the initial values of the parameters in eq. (3.30) and calculate the resulting values of $A(t)$ for $t < t^*$, where t^* denotes the time when the assumption of constancy of the parameters is not plausible any more. Then, we substitute

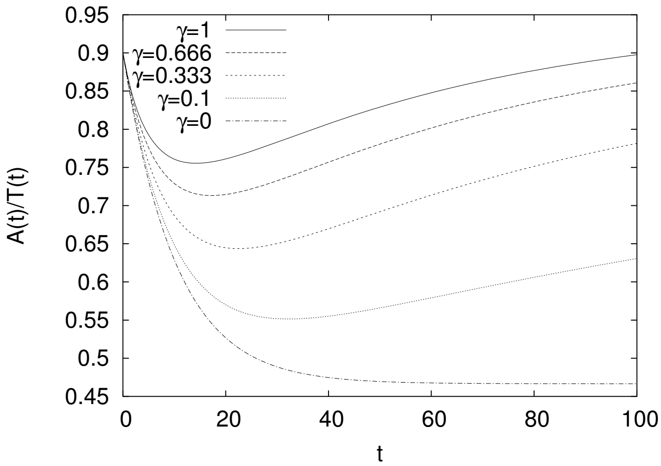
[3]The Nelson-Phelps equation can be solved analytically also if the factor $\frac{T(0)-A(0)}{A(0)}$ is replaced by $\frac{T(t)-A(t)}{A(t)}$.

new parameter values in eq. (3.30) (in particular, $A(t^*)$ and $T(t^*)$ instead of $A(0)$ and $T(0)$) and perform the analysis for the period $t^* \leq t \leq 2t^*$, etc. If $t^* \to 0$, we get an exact solution to the differential equation:

$$\dot{A}(t) = \xi A(t)^{1-\gamma} + d\frac{T(t) - A(t)}{A(t)}A(t), \tag{3.31}$$

i.e. eq. (3.29) with a variable diffusion term.

Figure 3.1: **Example of dynamics of the relative technology $A(t)/T(t)$ for various values of γ, with $t^* = 1$**



Explanation: parameter values: $\xi = 0.049$, $d = 0.044$, $A(0) = 0.9T(0)$, $\xi_0 = 0.1$ (for the technology leader).

An example of dynamics of the relative technology $A(t)/T(t)$ for various values of the parameter γ, with $t^* = 1$, is depicted in Fig. 3.1. The values of the remaining parameters are: $\xi = 0.049$, $d = 0.044$, $A(0) = 0.9T(0)$, $\xi_0 = 0.1$ (for the technology leader). According to the Benhabib-Spiegel model (subsection 1.3.1), the technology of a country with these parameter values will always grow slower than the technology of the technology leader (since $\xi + d < \xi_0$) and the limiting value of the ratio $A(t)/T(t)$ (at infinity) is 0. Setting $\gamma = 0$, we obtain the Nelson-Phelps model, which implies a constant decrease of the relative technology to $\Omega = 0.4665$ (eq. (1.78)). The values $\gamma > 0$ lead in turn to a reversal of the tendency of relative technology and it happens earlier for higher values of γ. For $\gamma < 0$, we have a constant decrease of the relative technology to zero in finite time or to the equalization of technologies in the given country and the technological leader, also in finite time.

3.2.3 Empirical analysis for the OECD countries

The empirical analysis was performed in the following way. We assumed that the relevant parameters of the model do not change within one-year periods (in particular this concerns the factor $\frac{A(0)-T(0)}{A(0)}$), thus $t^* = 1$ year. The period of the analysis was 1981-1999 – for this period we have all the necessary data on R&D expenditures for most countries [81][4]. For some of the countries we have the data only for the period 1991-1999. All technology

[4] See Appendix C.

levels are normalized such that USA in 1981 has unit technology. The value of the parameter γ was calibrated in such a way that the resulting dynamics for the United States was as close as possible to the empirical dynamics for this country (which is assumed to be the technological leader). The calibration criterion was the *Mean Relative Error* (MRE) of technology level estimate, defined for the i-th country as:

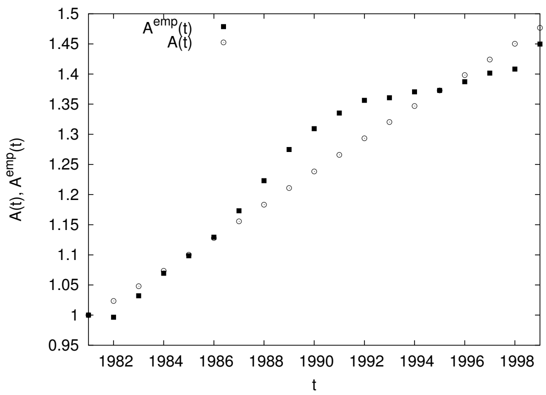
$$MRE = \sum_{t=1}^{\tau} \frac{|A_i(t) - A_i^{emp}(t)|}{A_i^{emp}(t)}, \tag{3.32}$$

where $A_i(t)$ denotes the level of technology that results from the model at time t, $A_i^{emp}(t)$ the empirical level of technology at time t, τ is the length of the analysis period (for most countries $\tau = 19$).

Fig. 3.2 shows the empirical ($A^{emp}(t)$) and resulting from the model ($A(t)$) levels of technology in USA, which correspond to the calibrated value $\gamma = 0.96$ (which minimizes the mean relative error (3.32)). The model implies that technology grows almost linearly (exactly linear behaviour corresponds to $\gamma = 1$ and constant R&D expenditures). We can observe large fluctuations in the empirical dynamics of $A^{emp}(t)$[5], which implies that the agreement of the model and empirical data is moderately good.

We analyze the model implications for the following γ values:

Figure 3.2: **Empirical ($A^{emp}(t)$) and resulting from the model ($A(t)$) levels of technology in USA for the calibrated value** $\gamma = 0.96$



Explanation: all values with respect to 1981.

- $\gamma = 5/4$,
- $\gamma = 1$ (linear growth of technology within one-year periods),
- $\gamma = 0.96$ (the calibrated value of γ),
- $\gamma = 3/4$,
- $\gamma = 1/2$,
- $\gamma = 1/4$,

[5]These fluctuations result from the method of calculation of technology levels A, which is sensitive to the cyclical fluctuations of GDP p.c.

- $\gamma = 0$ (the Nelson-Phelps model),
- $\gamma = -1$ (in this case the growth rate of technology increases with increasing technology).

Tab. 3.5 shows the calculated values of the diffusion coefficient for the analyzed countries. These are the values that minimize MRE for any given country.

The diffusion coefficients d differ substantially among the countries. The largest differences can be observed for high values of the parameter γ –

Table 3.5: **Diffusion coefficients d that minimize MRE for the analyzed countries**

		γ							
C'try	h'/h	1.25	1.00	0.96	0.5	0.50	0.25	0.00	-1.00
AUS	0.93	0.012	0.014	0.014	0.015	0.017	0.019	0.020	0.025
AUT	0.71	0.014	0.017	0.017	0.019	0.021	0.023	0.024	0.025
BEL	0.88	0.018	0.018	0.018	0.018	0.017	0.017	0.017	0.015
CAN	1.03	0.019	0.020	0.020	0.021	0.023	0.024	0.025	0.027
CZE	0.88	0.011	0.013	0.013	0.014	0.015	0.016	0.017	0.018
DEN	0.94	0.024	0.026	0.027	0.028	0.030	0.031	0.033	0.034
FIN	1.01	0.018	0.021	0.021	0.023	0.025	0.027	0.028	0.030
FRA	0.66	-0.012	-0.009	-0.009	-0.007	-0.004	-0.002	0.001	0.008
GER	0.95	-0.017	-0.013	-0.012	-0.010	-0.006	-0.003	0.000	0.010
GRE	0.74	0.015	0.016	0.016	0.016	0.017	0.017	0.017	0.018
HUN	0.67	-0.016	-0.014	-0.013	-0.012	-0.010	-0.009	-0.008	-0.005
IRL	0.84	0.043	0.045	0.045	0.046	0.048	0.049	0.051	0.052
ISL	0.75	0.066	0.066	0.066	0.066	0.065	0.064	0.063	0.056
ITA	0.64	0.042	0.044	0.044	0.045	0.047	0.048	0.048	0.048
JPN	0.81	-0.007	-0.001	0.000	0.004	0.008	0.012	0.015	0.024
KOR	0.96	0.003	0.008	0.009	0.012	0.015	0.018	0.020	0.022
MEX	0.61	0.006	0.007	0.007	0.008	0.008	0.008	0.008	0.008
NED	0.98	-0.006	-0.004	-0.003	-0.001	0.001	0.004	0.006	0.011
NZL	0.98	0.006	0.008	0.008	0.009	0.011	0.012	0.013	0.016
NOR	1.06	0.041	0.043	0.044	0.044	0.045	0.045	0.046	0.048
POL	0.57	-0.002	0.001	0.001	0.002	0.004	0.004	0.005	0.006
POR	0.41	0.015	0.016	0.017	0.017	0.018	0.019	0.019	0.020
SPA	0.64	0.014	0.015	0.015	0.016	0.017	0.018	0.019	0.021
SWE	1.14	-0.036	-0.033	-0.033	-0.030	-0.027	-0.024	-0.022	-0.012
SWI	0.85	-0.045	-0.044	-0.044	-0.043	-0.042	-0.040	-0.039	-0.033
TUR	0.45	-0.001	0.001	0.001	0.002	0.003	0.003	0.004	0.004
UK	0.88	-0.013	-0.010	-0.010	-0.007	-0.005	-0.002	0.000	0.008

Explanation: h'/h – human capital p.c. with respect to USA.
Source: own calculations based on Tabs. C.1 and C.2.

-0.045 to 0.066. The negative value of γ implies much smaller differences in d. According to Nelson and Phelps [61], the diffusion coefficient d is a function of human capital. To verify this hypothesis, we plot the calculated values of d vs. human capital resources p.c. computed from the Manuelli-Seshadri model – Figs. 3.3 ($\gamma = 0.96$), 3.4 ($\gamma = 0$) and 3.5 ($\gamma = -1$).

As can be seen on the plots, the Nelson-Phelps hypothesis should be rejected. For example, Sweden has the highest human capital resources p.c.

3

Figure 3.3: **The diffusion coefficients vs. human capital resources p.c. (with respect to USA) from the Manuelli-Seshadri model;** $\gamma = 0.96$

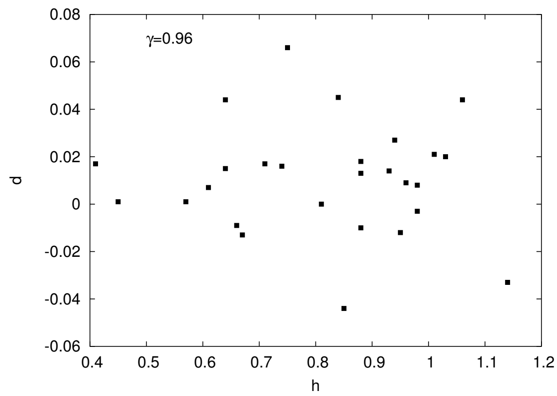


Figure 3.4: **The diffusion coefficients vs. human capital resources p.c. (with respect to USA) from the Manuelli-Seshadri model;** $\gamma = 0$

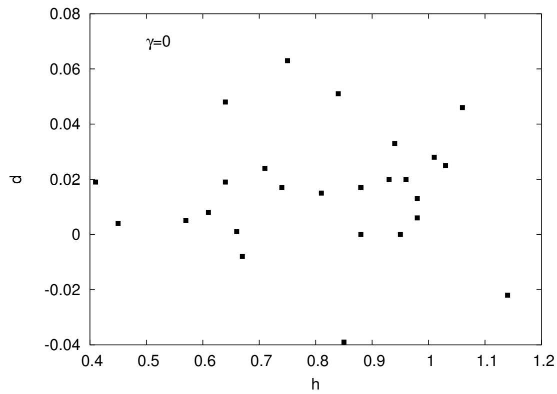
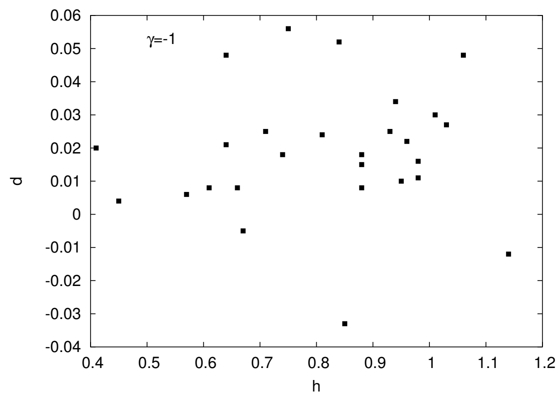


Figure 3.5: **The diffusion coefficients vs. human capital resources p.c. (with respect to USA) from the Manuelli-Seshadri model;** $\gamma = -1$



and a negative diffusion coefficient, whereas Italy has one of the smallest human capital resources p.c. and the diffusion coefficient is one of the highest.

Now, let us consider the negative values of diffusion coefficients for some countries. By analyzing the time series of R&D expenditures in such countries (Tabs. C.1 and C.2), one can notice that these expenditures are almost as high or sometimes even higher than the expenditures of the technological leader (USA). At the same time, their growth rates of technology are similar to the American growth rate. A negative value of d means thus that the R&D expenditures in such countries are allocated less effectively than in USA. It seems therefore that we should rewrite the technology dynamics equation as:

$$\dot{A}(t) = \varepsilon\xi A(t)^{1-\gamma} + d\frac{T(0) - A(0)}{A(0)}A(t), \qquad (3.33)$$

where the parameter ε denotes an average efficiency of 1% of GDP p.c. allocated to research and development. The values $\varepsilon < 1$ lead to higher values of the diffusion coefficient. However, to find an empirical measure for the parameter ε seems to be very difficult, if possible.

The problem of efficiency of R&D expenditures is not limited to the countries with high levels of these expenditures. The efficiency of allocation in the countries with lower expenditures can also be lower than in USA. Then, the diffusion coefficients in Tab. 3.5 would also be too low (their positive values mean, however, that the diffusion effect is much stronger than the efficiency effect, the reverse holds true for countries with $d < 0$). Thus, we should treat the calculated values of the diffusion coefficients as lower bounds – since one should not expect that the efficiency of allocation in these countries is higher than in USA. It is worth to mention that the γ values which are lower than the calibrated value of 0.96 can be interpreted as representing the model with $\gamma = 0.96$ and with the efficiency parameter ε proportional to A. For example, for the $\gamma = -1$ model, we can write:

$$\dot{A}(t) = A(t)^{1.96}\xi A(t)^{1-0.96} + d\frac{T(0) - A(0)}{A(0)}A(t), \qquad (3.34)$$

where $\varepsilon = A^{1.96}$ expresses the dependence of the efficiency of R&D expenditures on technology. Thus, this efficiency is the highest in countries with technology levels similar to the American level. The values of d at $\gamma = -1$ are higher for almost all countries with respect to the values for positive d, which is in accordance with the above given argumentation. However, the assumption that the efficiency parameter ε depends only on technology level is certainly too simplified.

Tab. 3.6 gathers the average values of the diffusion coefficient d for various values of γ. We calculated 3 kinds of averages:

- avg. 1 – arithmetic mean of all diffusion coefficients,
- avg. 2 – arithmetic mean of the positive diffusion coefficients, the negative values were set to 0,
- avg. 3 – arithmetic mean of the positive diffusion coefficients, the negative values were discarded,

As stated before, the values of the diffusion coefficients in Tab. 3.5 are lower bounds for the actual values. Thus, avg. 1 is the lower bound for the average value of d. This average is positive, which suggests that the diffusion effects are an empirical fact and they should be taken into account in empirical research on economic growth. The average value of the diffusion coefficient is between 0.8% and 1.9%, which means that the technology of a country with initially 50% worse technology than the leader will grow at a rate that exceeds the leader's rate by 0.8% to 1.9%, provided that R&D expenditures are the same.

Table 3.6: **Averages of the diffusion coefficients d for various values of γ**

	γ							
	1.25	1.00	0.96	0.75	0.50	0.25	0.00	-1.00
avg. 1	0.008	0.010	0.010	0.012	0.013	0.015	0.016	0.019
avg. 2	0.014	0.015	0.015	0.016	0.017	0.018	0.018	0.021
avg. 3	0.022	0.020	0.019	0.020	0.022	0.023	0.021	0.023

Source: own calculations based on Tab. 3.5.

Taking the lower bound for the diffusion coefficient to be 0, avg. 2 can be interpreted as a more realistic assessment of the average diffusion coefficient.

For countries for which the efficiency effect discussed above is relatively small (e.g. for small R&D expenditures the meaning of the diffusion term is much more important) avg. 3 can be regarded as the most realistic estimate for the average diffusion coefficient. For this average, the dispersion with respect to γ is rather small – from 1.9% to 2.3%. These values can be interpreted in a similar way to the values for avg. 1.

3.2.4 Empirical verification of the Manuelli-Seshadri model with technology diffusion

In this subsection we will present the results of empirical verification of the Manuelli-Seshadri model with the technology given by the analyzed model with technology diffusion. We will analyze the mean relative error (MRE) of GDP p.c. estimates, calculated from eq. (3.14), with the assumption that the diffusion coefficient has the same value for all countries. Such assumption is compatible with the result from the previous subsection – the diffusion coefficients d do not depend on human capital resources.

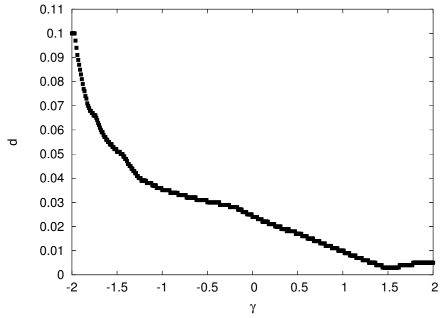
Tab. 3.7 presents the calculated values of the diffusion coefficient $\tilde{d}$ that minimize MRE of GDP p.c. estimates for 27 OECD countries. Fig. 3.6 shows the values of $\tilde{d}$ for arbitrary values of $\gamma \in [-2, 2]$. The lower the value of γ, the bigger the technology diffusion effect, since lower values of γ imply a slower growth of technology related to R&D expenditures. The obtained results are much better than the results obtained without taking technology into account (Tab. 2.10 (IALS countries) and 2.11 (the remaining OECD countries)) and comparable to the ones from the Manuelli-Seshadri model with exogenous technological progress (Tab. 3.3 (IALS countries) and 3.4

Table 3.7: **The values of the diffusion coefficient $\tilde{d}$ that minimize the mean relative error of GDP p.c. estimates in the Manuelli-Seshadri model with technological progress and technology diffusion**

	γ							
	1.25	1.00	0.96	0.5	0.50	0.25	0.00	-1.00
$\tilde{d}$	0.006	0.010	0.010	0.013	0.017	0.020	0.024	0.035
$MRE(\tilde{d})$ (%)	19.89	19.47	19.40	19.07	18.71	18.38	18.08	17.57
$MRE(d = 0)$ (%)	20.90	21.13	21.16	21.49	22.07	22.97	24.48	33.46

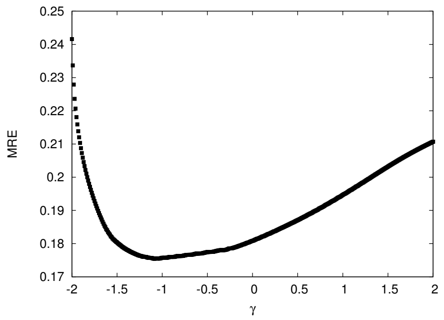
Explanation: $MRE(\tilde{d})$ – MRE for the calculated value of $\tilde{d}$, $MRE(d = 0)$ – MRE for the model without technology diffusion, i.e. for $d \equiv 0$ in all countries.

Figure 3.6: **The dependence of the diffusion coefficient $\tilde{d}$ that minimizes the mean relative error of GDP p.c. estimates in the Manuelli-Seshadri model with technological progress and technology diffusion on γ**



(the remaining OECD countries)), which takes into account the empirical level of technology of the analyzed countries in 1999. This suggests that the model of technology with a diffusion effect accounts in a satisfying way for the technology dynamics in the analyzed countries and allows to explain relatively high rates of technology growth in countries with relatively low R&D expenditures. Models without technology diffusion (e.g. the analyzed model with $d \equiv 0$ for all countries) lead to higher values of the mean relative error of GDP p.c. estimates, especially for lower values of γ. The values

Figure 3.7: **The dependence of MRE of the GDP p.c. estimate at the *calibrated* (ones that minimize $MRE(d)$) values of the diffusion coefficients $\tilde{d}$ on γ**



$\gamma \approx -1$ lead to the relatively smallest values of MRE, which is depicted in Fig. 3.7. At this value of γ the model without diffusion fails completely, leading to twice bigger errors of GDP p.c. estimates.

The main source of uncertainty in the analyzed model is the assumption of the same value of the diffusion coefficient for all countries. As shown in subsection 3.2.3, in some countries the R&D expenditures are similar or higher than in USA and their technology growth rates are similar to the American one, despite the fact that the diffusion effects should raise these growth rates above the American level. This leads to a conclusion that these countries must have a lower efficiency of R&D expenditures, which is not explicitly taken into account in the model. A simplified way to take this into account is to assume a lower value of the parameter γ. In such case, the diffusion coefficients increase and are closer to one another.

It seems that the above reasoning is an argument for taking lower values of γ, which is also suggested by Fig. 3.7.

However, we could obtain even better results if we introduced a measure of efficiency of R&D expenditures. Thus, the diffusion coefficients would be positive and would have similar values in different countries and γ could be assumed to equal 0.96, the value that gives the best fit to empirical data in the case of USA, the country for which $d \equiv 0$ by definition. Introduction of such measure of R&D efficiency could in this way be a direction for further research on this model.

3

3.3 Endogenous model of technological change with technology diffusion

In this section we will consider an *endogenous* model of technological change with technology diffusion and two components of technology. This is an extension of the Romer model.

3.3.1 Setup of the model

Let us consider an economy in which production at time t is given by the function:

$$Y(t) = (B(t)H_Y(t))^{1-\theta} \int_0^{\infty} x(i,t)^{\theta} di, \tag{3.35}$$

where $x(i)$ denotes the amount of i-th intermediate good, H_Y human capital resource employed in the production sector, B can be interpreted as the efficiency of implementation of the intermediate goods production technology. We assume that it is not enough to *invent* a technology, it also has to be *adapted* to be effectively used in the economy.

Let us assume that the range of available intermediate goods equals A, i.e. $\forall\, i > A: \;\; x(i) = 0$. Symmetry implies that $\forall\, i \leq A: \;\; x(i) = \bar{x}$. Hence, physical capital resource can be defined as:

$$K = \int_0^A x(i)di = A\bar{x}. \tag{3.36}$$

The production function can be written as:

$$\begin{aligned} Y &= (BH_Y)^{1-\theta} \int_0^A \bar{x}^{\theta} di = (BH_Y)^{1-\theta} A \bar{x}^{\theta} = \\ &= (BH_Y)^{1-\theta} A^{1-\theta} A^{\theta} \bar{x}^{\theta} = K^{\theta} (ABH_Y)^{1-\theta}. \end{aligned} \tag{3.37}$$

This is a Cobb-Douglas production function with Harrod-neutral technology, being a product of two functions – A and B, which represent the range of available intermediate goods and the efficiency of their use, respectively.

The form of eq. (3.37) implies that A and B are complementary technological factors. The development of technology and thus production increase with given K and H resources can result from intermediate goods production technology investments or from investments that raise the efficiency of application of prior technologies. Thus, we have two research sectors which aim at increasing the technology level of the analyzed country, i.e. increasing the product AB. The marginal rate of substitution between A and B equals:

$$MRS_{AB} \equiv \frac{\partial Y}{\partial A} : \frac{\partial Y}{\partial B} = \frac{B}{A}, \tag{3.38}$$

which means that one unit of A (technology of production of one intermediate good) can be replaced by B/A units of B (which is an increase of the efficiency of prior technologies by B/A units). In the case when one technological factor dominates over the other, its further development leads to relatively smaller increase in productivity – thus one can not continually introduce new technologies for intermediate goods without their adaptation to meet the specific requirements of a given economy. The reverse is also true – one can not constantly adapt old technologies – sometimes an introduction of a brand new technology is more profitable.

Let us assume that the technologies of intermediate goods production may come only from other countries – thus the increases in A can only come from technology diffusion. We will take the following form of the dynamics equation for the variable A:

$$\dot{A}(t) = d\frac{h_A(t)}{\tilde{h}_{AB}}\left(T(t) - A(t)\right), \tag{3.39}$$

where d is the technology diffusion parameter, $h_A(t)$ human capital p.c. (at time t) employed in the diffusion sector (i.e. the sector that imports the technologies of intermediate goods production from the technological leader), $\tilde{h}_{AB}$ the total amount of human capital p.c. used in both research sectors[6], and $T(t)$ is the range of intermediate goods available in the technological leader country.

We thus assume that the technological leader renders the technologies of intermediate goods production accessible to other countries the more willingly, the older (the more distant from the technology frontier) the technology is. The larger the stock of "research" human capital devoted to introducing new technologies, the faster the technology transfer. We also assume that the growth rate of technology is linear in the stock of human capital, i.e. each worker contributes the same to diffusion.

The variable $B(t)$ will be interpreted as the level of adjustment of intermediate goods production technologies to the characteristics of the economy of a given country (e.g. the economy's structure). We assume the following form of the $B(t)$ dynamics equation:

$$\dot{B}(t) = \xi(t)\left(\frac{h_B(t)}{\tilde{h}_{AB}}\right)^{\eta} B(t)^{1-\gamma}, \tag{3.40}$$

where ξ denotes the expenditures on technology implementation, $h_B(t)$ the amount of human capital in the implementation sector, and η and γ are parameters. The sum $h_A(t)+h_B(t)$ is denoted by $h_{AB}(t)$ and is given at any time t as $h_{AB}(t) = h(t) - h_Y(t)$, where $h(t)$ – the total amount of human capital p.c. in the economy, $h_Y(t)$ – its fraction used in the production sector.

We also assume that $\eta \in (0, 1)$, i.e. there are decreasing returns to scale in the implementation sector, since the work of some researches can lead to the same effects, i.e. to two inventions of the same way to effectively adapt some technology from the technological leader. Thus, the imitation sector's activity is more "creative" than the activity of the diffusion sector which does not suffer from decreasing returns to scale[7].

To effectively apply some technology i, it is necessary to incur costs related to its introduction (this cost is the production decrease due to the use of some human capital in the research sector) and implementation (ex-

[6] $\tilde{h}_{AB}$ can be taken as the total human capital p.c. in the research sectors at any arbitrary time – then the results of the empirical analysis can be a little different.

[7] In the general case, one can assume decreasing returns to scale also in the diffusion sector. The above argumentation suggests that the exponent of h_A in eq. (3.39) will be closer to 1 than the exponent η in eq. (3.40). Here, we consider the case in which $\dot{A}$

3

penditures ξ and the decreased human capital resource in the production sector).

In the technological leader country the level of technology $T(t)$ (the range of intermediate goods available at time t) grows exclusively because of the activity of the research and development sector (there is no diffusion). The dynamics of technology in the technological leader country is governed by the same type of equation as (3.40):

$$\dot{T}(t) = \xi(t)\left(\frac{h_T(t)}{\tilde{h}_T}\right)^{\eta} T(t)^{1-\gamma}, \tag{3.41}$$

where $h_T(t)$ denotes the total amount of research human capital p.c. (at time t) and $\tilde{h}_T$ its value in the last period (i.e. $\tilde{h}_T = \tilde{h}_{AB}^{USA}$).

3

We thus assume that the R&D sector's activity in the leader country is analogous to the implementation sector of other countries and depends on the expenditures ξ and human capital resources used in this sector (denoted by h_T; these resources can not be used in the production sector).

Now, we will find the solutions to the differential equations (3.39), (3.40) and (3.41). We assume that all quantities in these equations are constant, except for, respectively, A, B and T. We denote by $Y(s)$ the Laplace transform of the variable $A(t)$, thus eq. (3.39) takes the form:

$$sY(s) - A(0) = \frac{dh_A T}{\tilde{h}_{AB} s} - d\frac{h_A}{\tilde{h}_{AB}} Y(s). \tag{3.42}$$

Hence,

$$Y(s) = \frac{A(0)}{s + d\frac{h_A}{\tilde{h}_{AB}}} + \frac{dh_A T}{\tilde{h}_{AB} s(s + d\frac{h_A}{\tilde{h}_{AB}})}. \tag{3.43}$$

Taking the inverse Laplace transform yields:

$$A(t) = (A(0) - T(0))\, e^{-d\frac{h_A(0)}{\tilde{h}_{AB}}t} + T(0). \tag{3.44}$$

This equation is valid for a short period after time 0 (such that $T(t) \approx T(0)$ and $h_A(t)/\tilde{h}_{AB} \approx h_A(0)/\tilde{h}_{AB}$; it seems realistic to assume that the latter condition is exact, i.e. $h_A(t)/\tilde{h}_{AB} = h_A(0)/\tilde{h}_{AB}$ for small t – the employment structure can not be perfectly elastic – any movements between the sectors can not be immediate) – thus we assume that t is smaller than some constant t^*.

Introducing now an auxiliary variable $x \equiv B^{\gamma}$ and denoting by $X(s)$ the Laplace transform of the variable $x(t)$, we get the transformed eq. (3.40):

$$X(s) = \frac{x(0)}{s} + \frac{\gamma\xi\left(\frac{h_B}{\tilde{h}_{AB}}\right)^{\eta}}{s^2}. \tag{3.45}$$

Taking now the inverse Laplace transform and returning to the variable $B(t)$, we get:

$$B(t) = \left(B(0)^{\gamma} + \gamma\xi(0)\left(\frac{h_B(0)}{\tilde{h}_{AB}}\right)^{\eta} t\right)^{1/\gamma}. \tag{3.46}$$

Analogously,

$$T(t) = \left(T(0)^{\gamma} + \gamma\xi(0)\left(\frac{h_T(0)}{\tilde{h}_T}\right)^{\eta} t\right)^{1/\gamma}. \tag{3.47}$$

Equations (3.46) and (3.47), similarly to eq. (3.44), are valid for $t < t^*$.

3.3.2 Optimal allocation of human capital in the research sector

Under the notion of *optimal allocation of human capital in the research sector* we will understand such allocation of human capital resource $H - H_Y$ (i.e. human capital not used in production; H denotes the total human capital in the economy) between the *diffusion sector* and the *implementation sector* that the growth rate of production p.c. $\dot{y}/y$ is maximal.

Production p.c. is given by:

$$y = k^{\theta}(ABh_Y)^{1-\theta}, \tag{3.48}$$

where: k – physical capital p.c., h_Y – production sector's human capital p.c. The growth rate of y equals:

$$\frac{\dot{y}}{y} = \theta\frac{\dot{k}}{k} + (1-\theta)\frac{\dot{A}}{A} + (1-\theta)\frac{\dot{B}}{B} + (1-\theta)\frac{\dot{h_Y}}{h_Y}. \tag{3.49}$$

The growth rate (3.49) can be written as:

$$\frac{\dot{y}}{y} = f(h_A, h_B) + const, \tag{3.50}$$

where:

$$f(h_A, h_B) = (1-\theta)\,d\frac{h_A}{\tilde{h}_{AB}}\frac{T-A}{A} + (1-\theta)\,\xi\left(\frac{h_B}{\tilde{h}_{AB}}\right)^{\eta}B^{-\gamma}, \tag{3.51}$$

and the term:

$$const = \theta\frac{\dot{k}}{k} + (1-\theta)\frac{\dot{h_Y}}{h_Y} \tag{3.52}$$

does not depend on h_A and h_B[8]. Since $h_A = h - h_Y - h_B$, the expression (3.51) can be written as a function of only one variable h_B:

$$f(h_B) = (1-\theta)\,d\frac{(h-h_Y-h_B)}{\tilde{h}_{AB}}\frac{(T-A)}{A} + (1-\theta)\,\xi\left(\frac{h_B}{\tilde{h}_{AB}}\right)^{\eta}B^{-\gamma}. \tag{3.53}$$

The growth rate $\dot{y}/y$ will be maximal (for given $\dot{k}/k$ and $\dot{h_Y}/h_Y$) when the function $f(h_B)$ reaches a maximum. The necessary condition for the existence of an extremum of a function of one variable $f'(h_B) = 0$ implies that the implementation sector's human capital has to satisfy:

$$h_B^* = \tilde{h}_{AB}\left(\frac{d}{\eta\xi}B^{\gamma}\frac{T-A}{A}\right)^{\frac{1}{\eta-1}}. \tag{3.54}$$

The second derivative of the function $f(h_B)$ with respect to h_B equals:

$$f''(h_B) = (1-\theta)\xi\frac{\eta(\eta-1)}{\tilde{h}_{AB}^{\eta}}h_B^{\eta-2}B^{-\gamma}. \tag{3.55}$$

Since $\eta \in (0,1)$, $f''(h_B) < 0$ for arbitrary h_B. The value h_B^* corresponds thus to the maximum of the function $f(h_B)$. Hence, the optimal allocation of human capital p.c. satisfies for the diffusion sector:

[8] An alternative formulation in which h_A, h_B and h_Y can take any values, only limited by the balance condition $h_A + h_B + h_Y = h$ (with given h), i.e. the problem of allocation of human capital between three sectors to maximize $\dot{y}/y$, leads, for realistic values of the parameters, to unrealistically small values of h_Y. The criterion for the maximization of $\dot{y}/y$ is thus improper for such allocation problem.

$$h_A^* = h - h_Y - \tilde{h}_{AB}\left(\frac{d}{\eta\xi}B^{\gamma}\frac{T-A}{A}\right)^{\frac{1}{\eta-1}}. \tag{3.56}$$

3.3.3 Empirical analysis for the OECD countries

The empirical analysis based on this model was performed in a similar fashion to the analysis of the model from section 3.2. We assume $t^* = 1$ year and again cover the period 1981-1999, for which we have the necessary data on R&D expenditures for most countries[9] [81] (the parameter ξ in the dynamics equations for variables B and T) and the number of researchers per 1000 employed [82], [83].

The number of researchers per 1000 employed was used to find the amount of human capital that is used in the research sectors – h_{AB}. For USA, the technological leader which has no diffusion sector (by definition), the human capital resource in the (only) research sector is denoted by h_T and the symbol T is the level of technology that corresponds to the product AB for other countries. We assume that a person employed in the research sectors has three times larger amount of human capital than an average inhabitant of the respective country[10].

The total amount of human capital for 1999 has been found from the Manuelli-Seshadri model (see Tabs. 2.9 and 2.11 for the human capital resources p.c. with respect to USA). We assume that in the preceding years it decreases with an average rate of 1% per year[11]. All technology levels are given with respect to USA in 1981. We assume that for each country $A = B$ holds in 1981, i.e.:

$$A(0) = B(0) = \sqrt{A_{emp}(0)}. \tag{3.57}$$

We consider a few values of the parameter η:

- $\eta = 0.01$ (the growth rate of B is thus almost indpendent on h_B),
- $\eta = 0.19$ (this value leads to the smallest mean absolute error of GDP p.c. estimation in the Manuelli-Seshadri model, with analogous procedure of analysis to the one in subsection 3.2.4. The mean absolute error is then 18.96%.),
- $\eta = 0.25$,
- $\eta = 0.5$,
- $\eta = 0.75$,
- $\eta = 0.99$ (almost linear dependence of $\dot{B}$ on h_B).

The parameter γ for a given value of η has been calibrated in such a way that the model and empirical dynamics for the technological leader were as close as possible, with the mean relative error as the matching criterion.

The results of the calibration procedure for the parameter γ are shown in Fig. 3.8. Increasing η leads to lower calibrated values of γ, except for the interval $\eta \in (0.3; 0.4)$, where we have a reversed dependence resulting from irregularities in the time series of ξ, h_T and T.

Fig. 3.9 shows the empirical ($T_{emp}(t)$) and model ($T(t)$) technology level for USA, corresponding to the calibrated value of $\gamma = 0.61$ which minimizes the mean relative error of GDP p.c. estimation in the Manuelli-Seshadri model with technology given by the presently analyzed model. The impli-

[9] See Appendix C.

[10] The case of other ratios of human capital of a researcher to human capital of an average inhabitant has also been investigated and it leads to very similar conclusions.

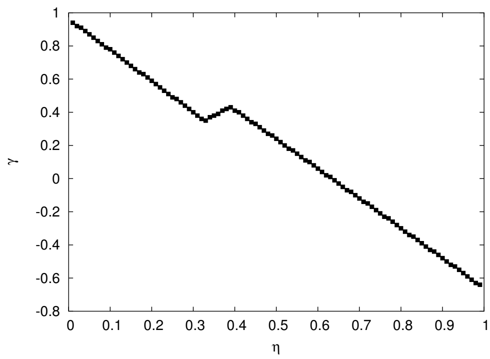
cation of the model is that technology grows almost linearly (strictly linear behaviour would be observed for $\gamma = 1$, $\eta = 1$ and constant R&D expenditures and human capital resources h_T). The fluctuations in the empirical dynamics of $T^{emp}(t)$ imply that the fit of the model to empirical data is moderately good.

Table 3.8: **The diffusion coefficients d that lead to the minimal MREs**

$\gamma =$	-0.64	-0.21	0.24	0.49	0.61	0.94
Country $\eta =$	0.99	0.75	0.50	0.25	0.19	0.01
AUS	0.217	0.206	0.168	0.111	0.096	0.049
AUT	0.415	0.391	0.318	0.210	0.183	0.055
BEL	0.222	0.206	0.152	0.091	0.078	0.035
CAN	0.444	0.397	0.237	0.126	0.108	0.051
CZE	0.088	0.085	0.076	0.058	0.053	0.030
DEN	0.527	0.491	0.397	0.258	0.217	0.099
FIN	0.687	0.667	0.621	0.360	0.276	0.107
FRA	0.334	0.292	0.081	0	0	0
GER	0.216	0.154	0.066	0.027	0.020	0
GRE	0.169	0.165	0.161	0.150	0.145	0.119
HUN	0	0	0	0	0	0
IRL	0.445	0.438	0.432	0.412	0.386	0.301
ISL	0.999	0.875	0.606	0.328	0.281	0.153
ITA	0.334	0.318	0.261	0.186	0.163	0.091
JPN	0.346	0.324	0.252	0.115	0.079	0.020
KOR	0.098	0.096	0.089	0.072	0.065	0.038
MEX	0.029	0.029	0.029	0.027	0.027	0.023
NED	0.176	0.148	0.089	0.045	0.037	0.010
NZL	0.128	0.124	0.109	0.068	0.060	0.031
NOR	0.587	0.584	0.444	0.319	0.244	0.115
POL	0.024	0.023	0.023	0.019	0.018	0.011
POR	0.210	0.206	0.200	0.184	0.181	0.155
SPA	0.162	0.159	0.153	0.137	0.130	0.085
SWE	0.256	0.181	0	0	0	0
SWI	0.380	0.097	0	0	0	0
TUR	0.020	0.019	0.019	0.017	0.016	0.011
UK	0.191	0.150	0.059	0.023	0.017	0

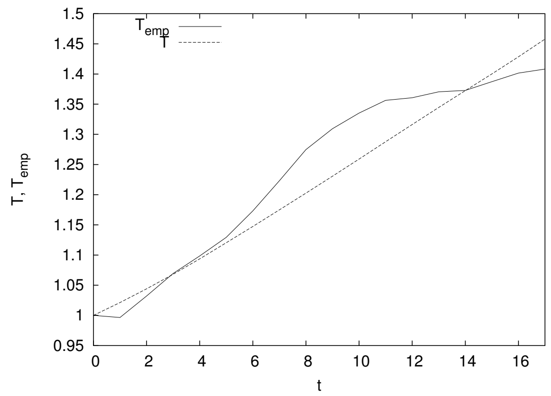
Source: own calculations based on Tabs. C.1, C.2, C.5 and C.6.

Figure 3.8: **The calibration of the parameter γ for a given value of η**



[11]The calculated human capital resources p.c. in the research sectors are given in Appendix C.

Figure 3.9: **Empirical ($T_{emp}(t)$) and model ($T(t)$) technology level for USA, with $\eta = 0.19$ and the calibrated value of $\gamma = 0.61$**



In Tab. 3.8 we present the values of the diffusion coefficient which minimize the mean relative error MRE, here defined as:

$$MRE = \sum_{t=1}^{\tau} \frac{|A_i(t)B_i(t) - A_i^{emp}(t)|}{A_i^{emp}(t)}, \tag{3.58}$$

Table 3.9: **The average values of the diffusion coefficients d at various values of η and γ**

$\gamma =$	-0.64	-0.21	0.24	0.49	0.61	0.94
$\eta =$	0.99	0.75	0.50	0.25	0.19	0.01
avg. 1	0.285	0.253	0.187	0.124	0.107	0.059
avg. 2	0.296	0.263	0.210	0.145	0.125	0.076

Source: own calculations based on Tab. 3.8.

where: $A_i(t)$ and $B_i(t)$ – technology levels resulting from the model at time t, $A_i^{emp}(t)$ – the empirical level of technology at time t, τ – length of the analysis period (for most countries $\tau = 19$, i.e. the period $0 =$ the year 1981, the period $\tau =$ the year 1999).

Similarly to the model analyzed in section 3.2, the differences in the calculated diffusion coefficients are very large. We get much higher values of these coefficients than in the previous model, where the form of the diffusion equation (3.39) is different. Thus, one can define an effective diffusion coefficient, which is comparable to the diffusion coefficient in the model of section 3.2:

$$d_{eff} = d\frac{h_A}{\tilde{h}_{AB}}. \tag{3.59}$$

In the countries where the diffusion sector is dominating, the effective diffusion coefficient d_{eff} is similar to the diffusion coefficient d. When the implementation sector dominates, there are large differences in the values of d_{eff} and d. Since $h_A/\tilde{h}_{AB}$ is not constant over time, the effective diffusion coefficient is also a function of time and thus we analyze it together with the time paths of h_A and h_B.

In the analyzed model, we also have zero values of the diffusion coefficients. They emerged in situations when decreasing d led to a decrease of MRE. The form of eq. (3.54) implies that negative values of d are not allowed, since $d = 0$ already means that all research human capital h_{AB} is employed in the implementation sector ($h_{AB} = h_B$). Analogously to the model from section 3.2, taking a lower efficiency of R&D expenditures ξ leads to lower efficiency of the implementation sector and thus makes the allocation of some human capital in the diffusion sector potentially profitable (in this way, $d > 0$).

Tab. 3.9 gathers the average values of the diffusion coefficient d for various values of parameters η and γ. We calculated two kinds of averages:

- avg. 1 – arithmetic mean of all diffusion coefficients,
- avg. 2 – arithmetic mean of positive diffusion coefficients, zero values were discarded.

At high η and low γ, we obtain the lowest MRE at high values of the diffusion coefficients – they are positive in almost all countries. Lowering η and increasing γ leads to an increased growth rate of the variable B, which means that the implementation sector becomes more effective, which in turn leads to lower values of the diffusion coefficients.

We will now analyze the time paths of the modelled quantities – the levels of technology A, B and their product AB, empirical technology levels A_{emp}, the effective diffusion coefficient d_{eff} and the division of human capital between the two research sectors – h_A and h_B. To simplify, we will call them *sector A* and *sector B*. All calculations were performed for $\eta = 0.19$ and $\gamma = 0.61$, the parameters that lead to the smallest MREs.

All the results are shown in Figs. 3.10-3.36.

Figure 3.10: **The model and empirical dynamics for Australia**

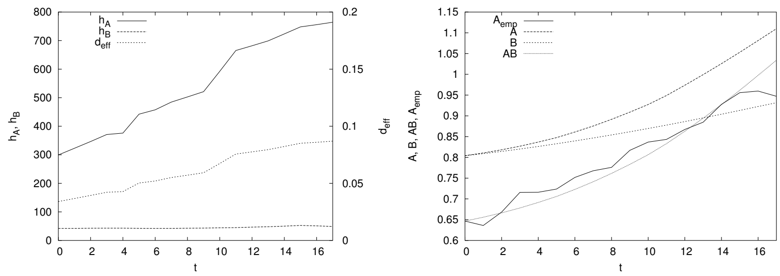


Figure 3.11: **The model and empirical dynamics for Austria**

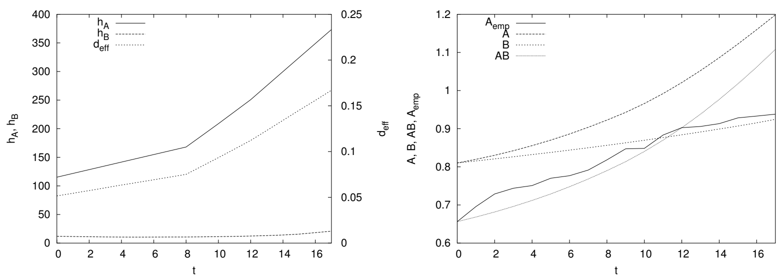


Figure 3.12: **The model and empirical dynamics for Belgium**

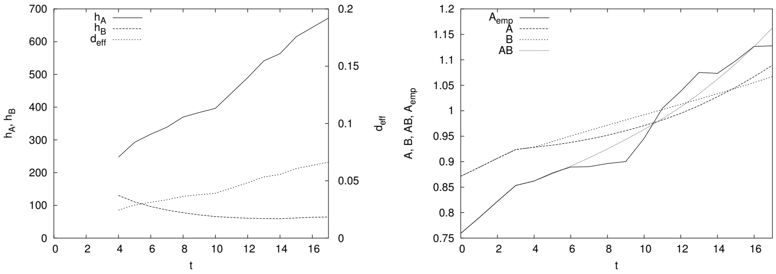


Figure 3.13: **The model and empirical dynamics for Canada**

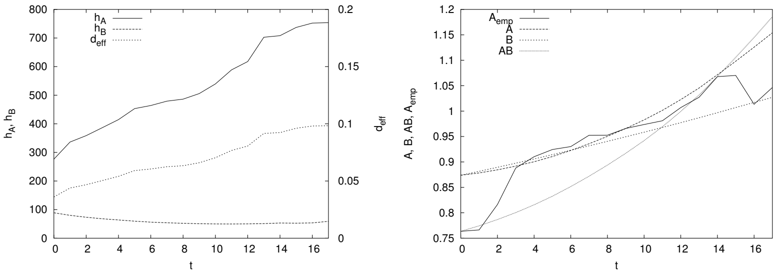


Figure 3.14: **The model and empirical dynamics for the Czech Republic**

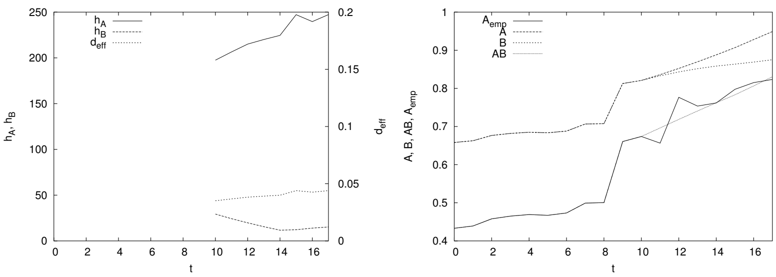


Figure 3.15: **The model and empirical dynamics for Denmark**

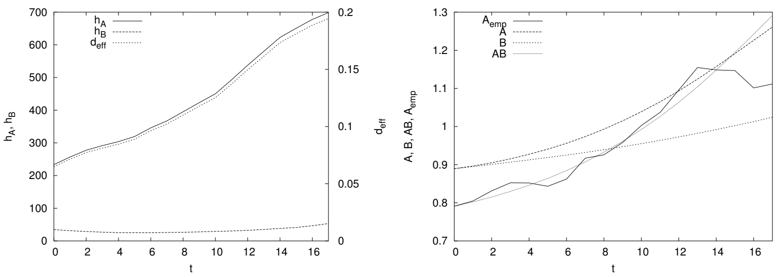


Figure 3.16: **The model and empirical dynamics for Finland**

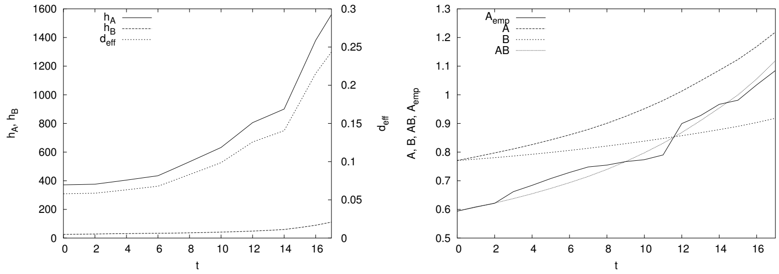


Figure 3.17: **The model and empirical dynamics for France**

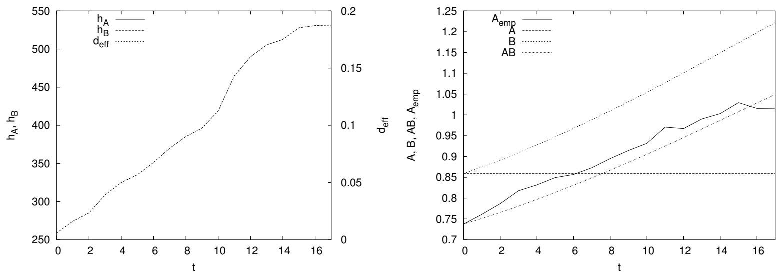


Figure 3.18: **The model and empirical dynamics for Germany**

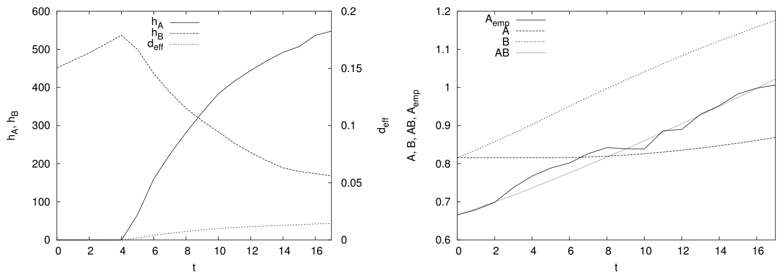


Figure 3.19: **The model and empirical dynamics for Greece**

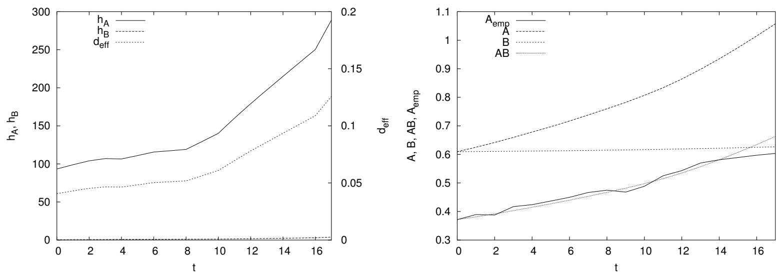


Figure 3.20: **The model and empirical dynamics for Hungary**

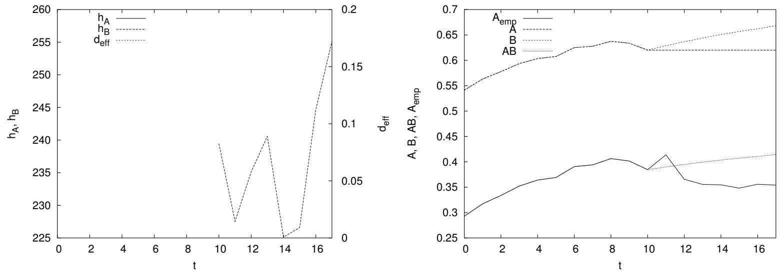


Figure 3.21: **The model and empirical dynamics for Ireland**

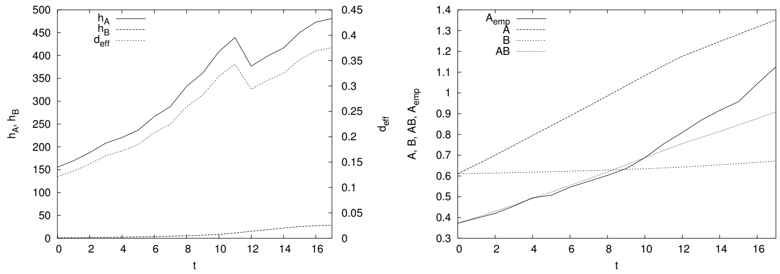


Figure 3.22: **The model and empirical dynamics for Iceland**

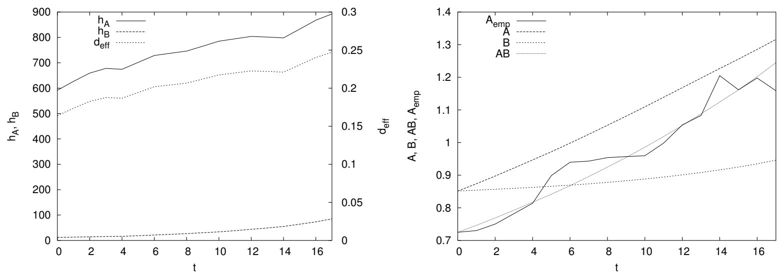


Figure 3.23: **The model and empirical dynamics for Italy**

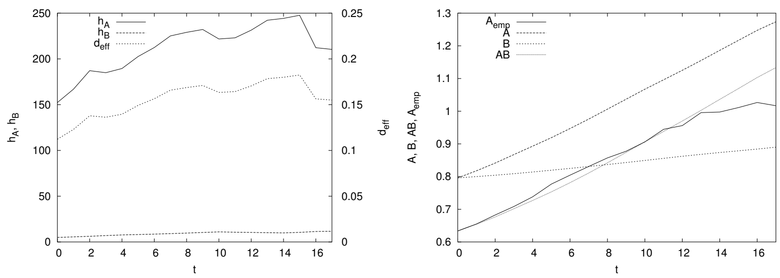


Figure 3.24: **The model and empirical dynamics for Japan**

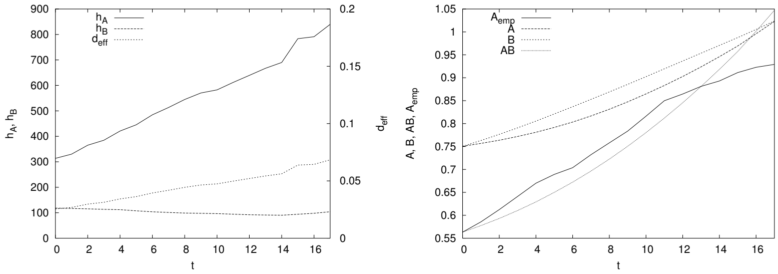


Figure 3.25: **The model and empirical dynamics for South Korea**

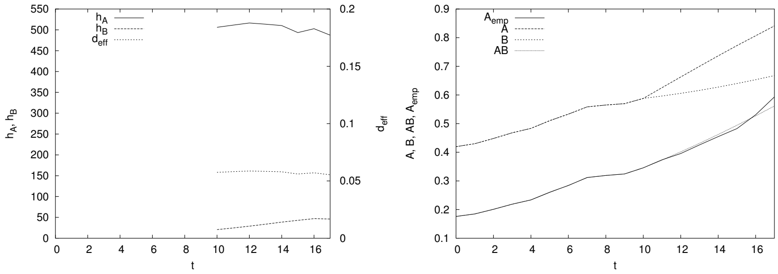


Figure 3.26: **The model and empirical dynamics for Mexico**

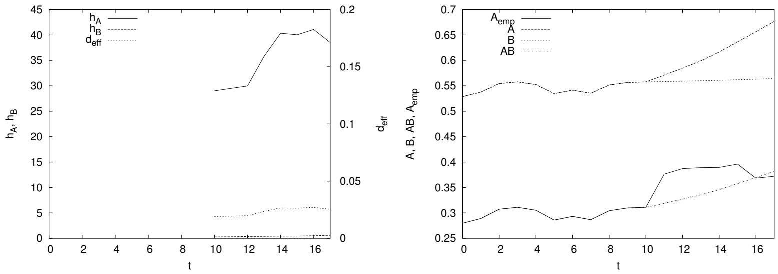


Figure 3.27: **The model and empirical dynamics for The Netherlands**

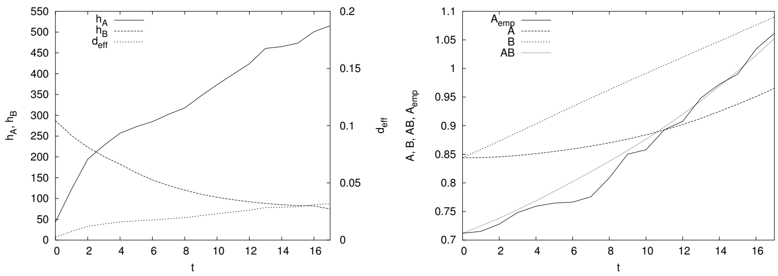


Figure 3.28: **The model and empirical dynamics for New Zealand**

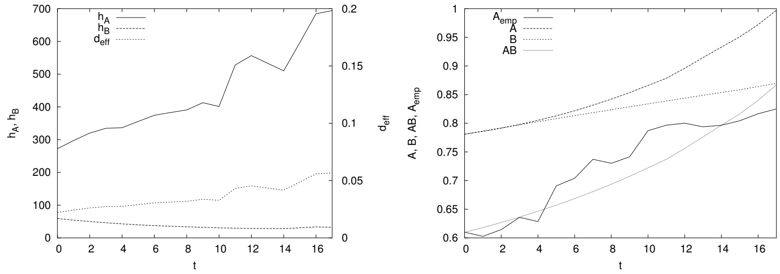


Figure 3.29: **The model and empirical dynamics for Norway**

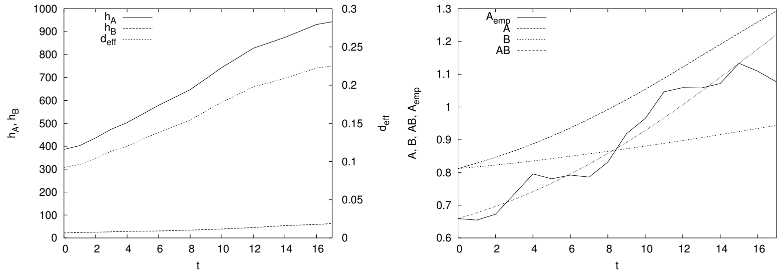


Figure 3.30: **The model and empirical dynamics for Poland**

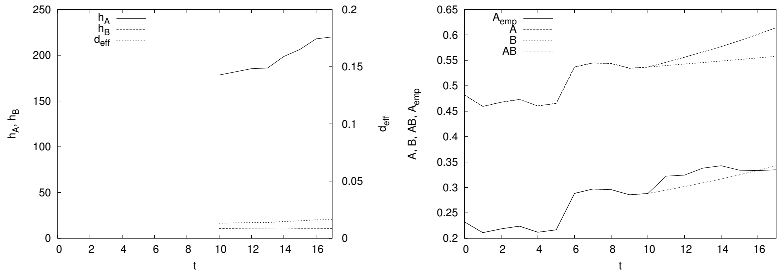


Figure 3.31: **The model and empirical dynamics for Portugal**

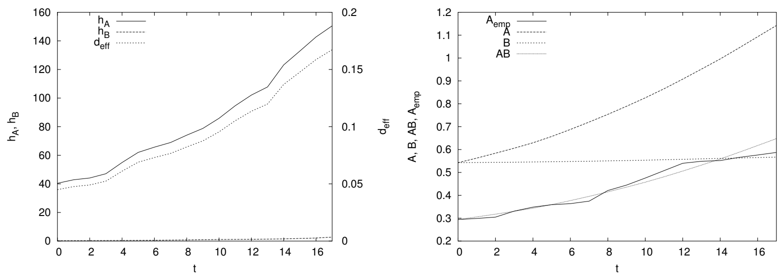


Figure 3.32: **The model and empirical dynamics for Spain**

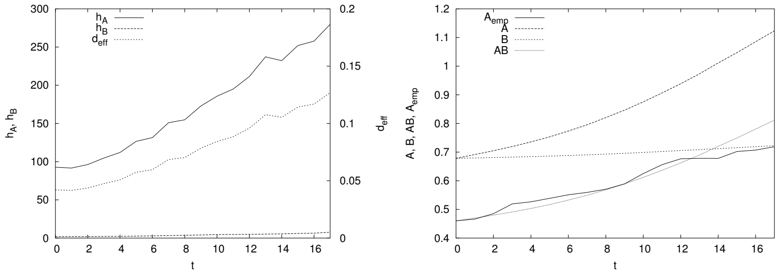


Figure 3.33: **The model and empirical dynamics for Sweden**

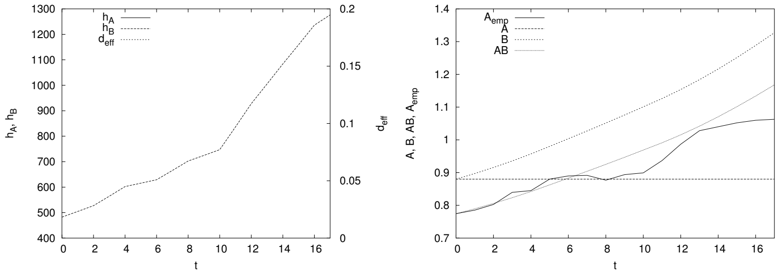


Figure 3.34: **The model and empirical dynamics for Switzerland**

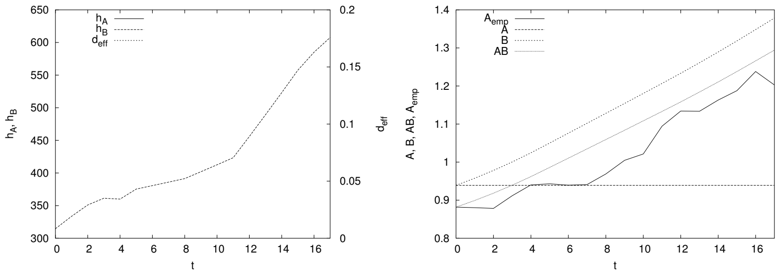


Figure 3.35: **The model and empirical dynamics for Turkey**

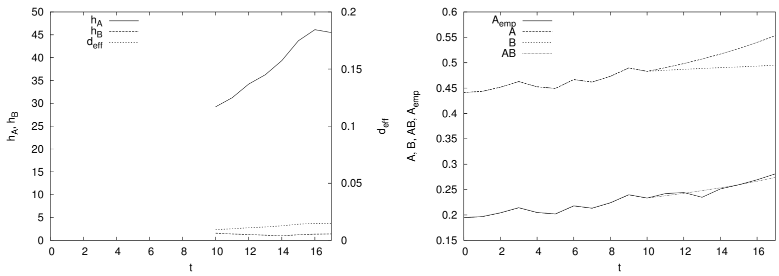
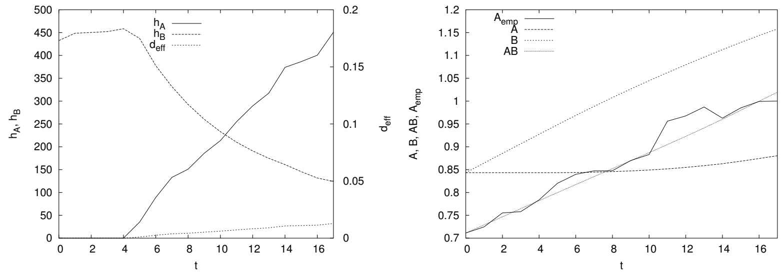


Figure 3.36: **The model and empirical dynamics for the United Kingdom**



In most countries the diffusion sector (A) dominates, which means that it employs a larger fraction of human capital than the implementation sector (B). These countries are: Australia, Austria, Belgium, Canada, the Czech Republic, Denmark, Finland, Greece, Ireland, Iceland, Italy, Japan, South Korea, Mexico, New Zealand, Norway, Poland, Portugal, Spain and Turkey. For these countries, we observe that the variable A grows faster than the variable B. One exception is Belgium, where a small initial excess of h_A with respect to h_B causes that both technological variables grow at a similar rate.

For a large group of the above-mentioned countries, the growing stock of human capital h_{AB} is equally divided between both sectors, i.e. h_A and h_B grow at a similar rate. Here we can include: Austria, Denmark, Finland, Greece, Ireland, Iceland, Norway, Portugal and Spain.

In South Korea, Mexico and particularly in Italy we observe in the recent years that the stock of h_A shrinks, while h_B increases, which can reverse the trend of diffusion-dominated growth.

In Australia, Poland and Turkey, the stock of human capital in sector B is almost constant, whereas h_A increases (except for the last period in Turkey). This means that the new research workers are employed only in the diffusion sector.

In Belgium, Canada, the Czech Republic, Japan and New Zealand we have a decreasing role of the implementation sector – the stock of h_B is going down, since new research workers are employed in sector A and, additionally, some workers move from B to A. This trend is, however, slowly reversing – in the last 1-2 years one can see some growth in h_B.

In most of the above-mentioned countries, the effective diffusion coefficients grow substantially, especially in Finland (from ca. 0.07 to ca. 0.25) and Portugal (from ca. 0.05 to ca. 0.17). The typical values of this coefficient are at the beginning of this period ca. 0.05 and ca. 0.10-0.15 in 1999. For the countries which had a high value of the diffusion coefficient in the model in section 3.2 (in particular Ireland and Iceland), the level of d_{eff} was high in the whole period of analysis, which confirms that technology diffusion was the most important mechanism of technological progress in these countries.

In Poland and Turkey in turn, the value of the effective diffusion coefficient was low in the whole period of analysis (ca. 0.02) and these are also the countries which had the lowest values of d in the model of subsection 3.2. Thus, despite the fact that technology diffusion is the dominating mechanism of technological progress in these countries, its efficiency is rather low.

For most countries, the mean relative errors are quite low – especially for Belgium, Finland, Iceland, the Czech Republic, South Korea and Turkey. In the last three countries, however, the period of analysis was restricted to the years 1991-1999.

The errors are the largest for Austria and Canada. For Austria, the model can not account for the significant decrease in the growth rate of A_{emp} in the recent years. For Canada, there is no clear explanation in the time series of ξ and h_{AB} for the fast growth of technology around 1983 and the decline of the growth rate in the recent years.

A second group of countries in the analysis is formed by France, Hungary, Sweden and Switzerland. In these countries, the diffusion coefficient is zero, which means that the efficiency of the diffusion process is zero and thus human capital is only allocated in the implementation sector. These countries in the model of section 3.2 had negative diffusion coefficients, hence we can conclude that technology diffusion is not the main mechanism of technological progress in these countries. However, the mean relative errors for them are not very low (except for France) – the reason for this is that fixed values of ξ and h_{AB} make it impossible to calibrate the growth of variable B. This is especially clear for Switzerland – the model-resulting curve AB lies above the empirical curve for all periods of our analysis. This confirms the conclusion from section 3.2 that these countries allocate their R&D expenditures less efficiently than USA – we would have much better agreement between the model and empirical time series for lower values of ξ. Then, the effective diffusion coefficients would be positive, which is more realistic – it seems improbable that the diffusion of technology is totally unimportant.

The third group of countries consists of Germany, The Netherlands and United Kingdom. In these countries, initially all the research human capital is employed in sector B. In the following periods, human capital moves from sector A to B and in 1984 (in The Netherlands) or around 1990 (Germany, United Kingdom) we observe that human capital stocks in both sectors are equal. Then, the diffusion sector starts to dominate, i.e. human capital resources in sector A begin to be larger than in sector B. Such dynamical behaviour of h_A and h_B results in faster growth of variable B with respect to variable A. The effective diffusion coefficient equals zero initially, but then it grows substantially. The mean relative errors for this group of countries are small.

As a summary, we can state that the empirical analysis confirms the conclusion that technology diffusion is the most important mechanism of technological progress in most of the countries. This means that it is profitable to allocate much human capital in the sector that imitates technological solutions of the leader and allocate much less in the implementation sector. However, it is not advisable to resign fully from the implementation activity – this would lead to a lower rate of technology growth, since technological solutions from the leader country have to be adjusted to the economies' characteristics and specifications.

The main limitation of the model is that it does not take into account the fact that the efficiency of R&D expenditures can vary between the countries. Introducing a measure of efficiency is difficult, but it could alter some of the conclusions about the division of human capital between the diffusion and implementation sectors. This would be especially important with respect to the second group of countries.

The complexity of the model forced us to make a few arbitrary assumptions in the empirical analysis. An important simplification was to assume that the variables $A(t)$ and $B(t)$ were initially equal. Also, the calibration procedure of the parameters η and γ was rather simple – the elasticity of the growth rate of technology with respect to human capital in USA does not have to be the same as the elasticity of the growth rate of the variable $B(t)$ with respect to human capital stock in the implementation sector in other countries.

Still, the main conclusions from the model seem to be interesting. A way to extend this model could be to introduce the efficiency of R&D expenditures, technology diffusion between other countries (not only between a given country and the technological leader) or basic research in all countries, not only in the leader country.

3.4 Summary

In this section we analyzed three models of technological progress.

From the first model, we concluded that the combined effects of human capital and technological progress can much better account for the differences in GDP p.c. levels among the OECD countries. Moreover, we found that the growth rate of technology is much higher in many countries than in USA and thus technology diffusion effects must be important.

In the second model, we examined the properties of the diffusion effects, using a generalized technology dynamics equation.

In the third model, we introduced two research sectors – the diffusion sector and the implementation sector. The second and the third model confirmed the conclusion that technology diffusion is the most important mechanism of technological progress in most of the countries.

We also pointed to the directions for future research on technology diffusion. We proposed to take into account the possibly different efficiency of R&D expenditures in different countries and also the possibility that technology flows not only between the leader and other countries, but also between other pairs of countries.

Chapter 4

Technological progress – simulational approach

In this chapter we will present an alternative approach to technological progress, using the Monte Carlo simulational method. This method allows to analyze a set of individualized (heterogeneous) agents, who obey some rules and thus imitate the behaviour of real economic agents.

4.1 Theoretical principles of the Monte Carlo method

The Monte Carlo simulational method is well-suited to analyze the dynamics of systems in which the changes are determined by probabilities and do not result only from the current state of the system [53]. The dynamics of such systems is thus not deterministic – it is stochastic and depends on the generated sequence of random numbers. This means that each simulation can lead to a slightly different outcome – all of the outcomes are, however, consistent within some statistical error.

The Monte Carlo method was first used in physics in the 1940s. The pioneers of this method were Enrico Fermi, Stanislaw Ulam, John von Neumann and Nicholas Metropolis, who worked at this time in Los Alamos, aiming at the construction of an atomic bomb. Some calculations related to this subject were impossible to be performed analytically, but they turned out to be relatively simple with the use of even the most straightforward version of the Monte Carlo method.

Since then, the Monte Carlo methods became much more advanced and many versions, tailored to specific problems, emerged. However, the simplest methods are still quite important, which is due to the ever-increasing growth in computing power – in many cases it is more profitable to use a simpler and more computer power consuming method than to elaborate some new version that can save just a fraction of computer time.

The most classic applications of the Monte Carlo method are: numerical integration[1], random walk and optimization problems.

We will describe the general idea of the Monte Carlo method basing on the example of numerical integration [53]. Let us assume that we have a function of one variable[2] and we want to calculate the integral over some interval from a to b. We begin with the construction of a rectangle with vertices $(a, 0)$, $(b, 0)$, (a, y_0), (b, y_0), where $y_0 \geq f(x)$ for any $x \in [a, b]$. Then, we generate a random sequence of N pairs (x_i, y_i), where $i = 1, \ldots, N$, the values x are drawn from a uniform distribution on the interval (a, b) and y from a uniform distribution on $(0, y_0)$. To perform the simulation in the proper way, the consecutive numbers from both sequences can not depend on

[1] The Monte Carlo method is from the practical point of view the only way to compute many-dimensional integrals.

[2] To simplify, we assume that the function is positively-valued on the interval of interest.

the previous numbers[3]. We then count how many of these points are located under the function plot (we denote this number by N_0). The approximate value of the integral is:

$$\int_a^b f(x)dx \approx \frac{N_0}{N} y_0(b-a), \tag{4.1}$$

i.e. it is the area of the rectangle, multiplied by the fraction of points that lie under the function plot.

The Monte Carlo simulation consisted in probing the state space (the rectangle with sides of length $b-a$ and y_0) and estimating the volume of space below the function plot. To have a reliable estimate of the integral, we must generate at least around 1000 points to obtain 1-2% precision in the one-dimensional case. Increasing the number of points will increase the precision – 10000 points will give typically a 0.1% error, 100000 points 0.01% etc. This example shows two important aspects of Monte Carlo simulations. First, the number of points has to be quite large. Second, one must have a reliable random number generator, since the probability of generating each point in the rectangle has to be the same.

4

In all Monte Carlo simulations, the general rules are similar – one constructs the relevant state space and probes its properties with random numbers. The probability of generating points in this space does not have to, however, be equal for all points. Later, we will often use numbers from the normal distribution[4]. By *uniform distribution* we will understand from now on the uniform distribution on the interval $(0, 1)$.

4.2 A simple model of technology diffusion

In this section we will analyze a simple model of technological progress in a system (an economy), in which technology diffusion plays the key role. The model was introduced by Llas *et al.* in 2003 [55].

4.2.1 Setup of the model

Let us consider a one-dimensional lattice with N sites. At each site i, there is an economic agent, characterized by a single variable $A_i(t)$, which can be identified with the level of technology at some time t [5]. At every time step, one of the agents comes up with an innovation, which can be interpreted as an autonomous invention or an imitation of some outside technology (technology diffusion from outside). When this agent introduces an innovation that increases their technology, their neighbours become aware of this fact and if the innovation scale is large enough, they imitate this technology, i.e. they also introduce it. It is profitable for them if their improvement in technology is larger than a given parameter C, which measures the cost of imitation – thus the parameter C controls the imitation process in this economy. The constant innovating activity keeps the system out of equilibrium and enables long-term technological progress.

We will perform the Monte Carlo simulations in the following way. We choose a value of the parameter C and an initial distribution of the technological variable $A_i(0) = A^0$ for every i.

[3] In Appendix D we shortly discuss the method of random number generation.

[4] The method to generate numbers from the normal distribution is given in Appendix D.

[5] Generally, $A_i(t)$ can be any characteristics that other agents may want to imitate.

1. At given time t (Monte Carlo step), we randomly choose some agent i and his technology A_i grows by Δ (they innovate):

$$A_i \rightarrow A_i + \Delta, \tag{4.2}$$

where Δ is a random number from a uniform distribution on the interval $(0, 1)$.

2. Agents $j \in \Gamma(i)$, where $\Gamma(i)$ is the set of i-th agent's nearest neighbours, imitate i-th agent's technology if:

$$A_i - A_j \geq C, \tag{4.3}$$

i.e. if this condition holds, $A_j \rightarrow A_i$ (*inner* technology diffusion).

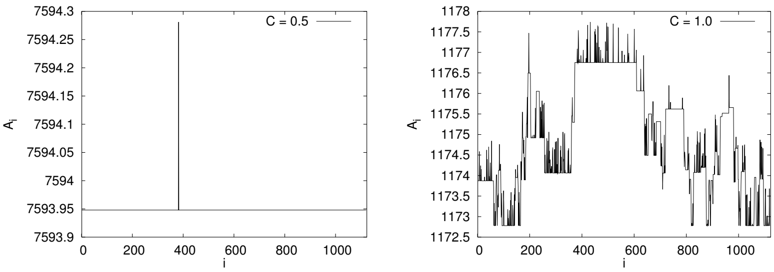
3. The imitation procedure (point 2) is repeated until some agent does not want to imitate the neighbour's technology. This finishes the Monte Carlo step.

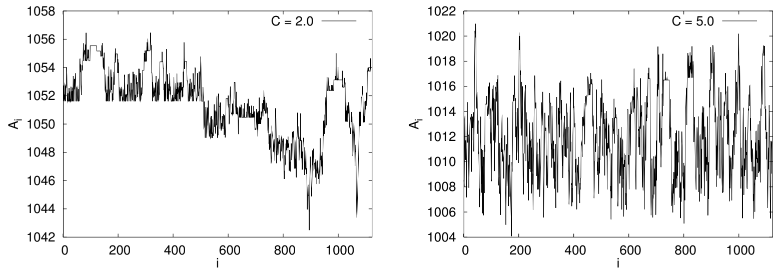
4.2.2 Properties of the model

The properties of the model depend on the value of the parameter C. Here we present the results for the following values: $C = \{0, 5; 1; 2; 5\}$.

Fig. 4.1 presents the levels of technology of 1123 economic agents after $18 \cdot 1123$ Monte Carlo steps[6] (thus, every agent innovates 18 times on average). The average technology level without inner technology diffusion would equal the sum of the initial technology level (1000) and the average number of innovations per agent multiplied by the average size of an innovation (0.5). For the considered system it would therefore be around 1009. The maximal number of imitations equals $1123 \cdot 18 \cdot 1122 = 22680108$ (every innovation can be imitated by the remaining 1122 agents) Low values of C mean that it is profitable to imitate almost every innovation and the levels of technology of almost all the agents is identical. For $C = 0.5$ after $1123 \cdot 18$ Monte Carlo steps, the average level of technology is ca. 7.5 times the initial level and results from ca. 40% of the maximum number of imitations. For $C = 1$, a single innovation does not result in an imitation (only ca. 0.35% of the maximum number of imitations takes place), which leads to irregularities in the system. These irregularities are not, however, very large and their scale is just a few agents. One can still see the traces of imitation *avalanches*, which cease to be noticeable for C more than around 5. For this value of C technological progress consists only in technology diffusion from *outside* or in independent innovations, which do not spread over the entire economy. The *inner* technology diffusion is almost absent – only 575 imitations took place in the simulation (0.0025% of the maximum number).

Figure 4.1: **Technology levels of 1123 economic agents after** $18 \cdot 1123$ **Monte Carlo steps (for** $C = \{0.5; 1; 2; 5\}$**)**





Thus, the parameter C determines the number of imitations and hence technological progress rate. This is illustrated in Figs. 4.2 and 4.3. The lower the value of C, the higher the growth rate of technology, since the flow of information between the agents is facilitated. The relatively small size of each innovation (between 0 and 1) prompts to interpret them as small improvements of the technological processes. These improvements can come from outside (an agent imitates some technological solution from another country) or be the result of their autonomous research activity. Thus, the model describes the effects of technology diffusion from outside and between the agents (the lower the value of C, the easier the inner technology diffusion).

Fig. 4.4 shows the dynamics of average technology for a few values of the parameter C. For low values of C, there is a steady growth of technology, since almost every innovation is imitated by almost all agents. High values

Figure 4.2: **The dependence of the average technology (A_{avg}) after $18 \cdot 1123$ Monte Carlo steps on the parameter C**

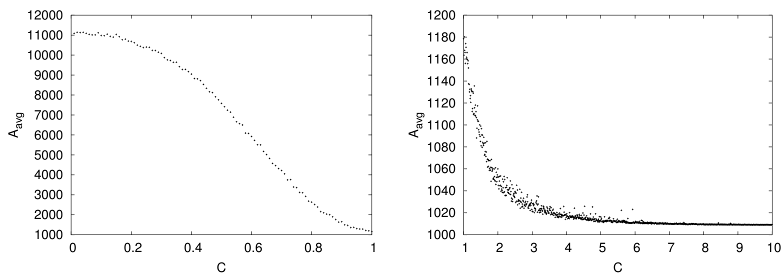
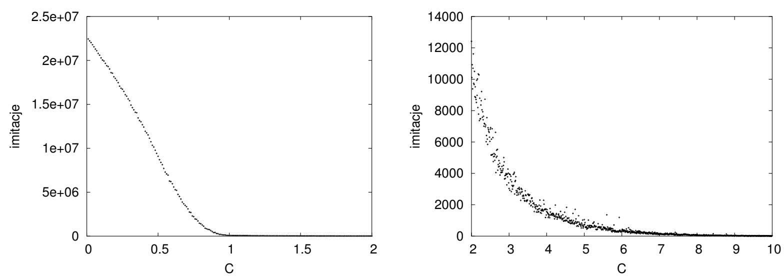


Figure 4.3: **The dependence of the number of imitations on the parameter C**



[6] There are 18 one-year periods in the empirical analysis of the next subsection; 1123 is the size of the system for USA.

Figure 4.4: **Dynamics of technology**

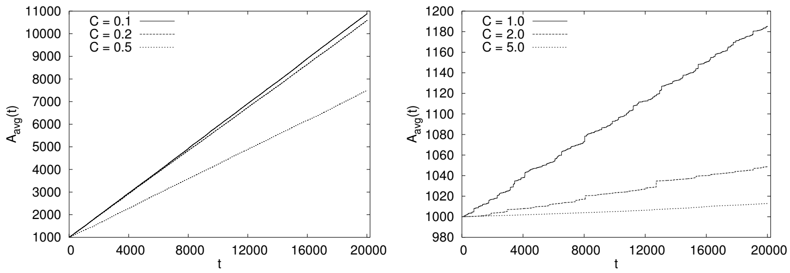
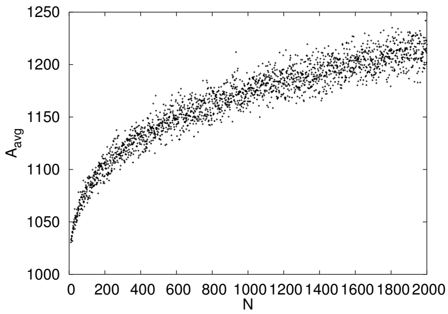


Figure 4.5: **The dependence of the average technology after** $18N$ **Monte Carlo steps on the system size** N**, for** $C = 1$



of C, in turn, imply technology jumps, i.e. periods of stability of technology are intertwined with periods of its sudden growth. This results from the fact that imitation takes place only after accumulating the effects of a few successive innovations.

We should emphasize that regardless of the value of the parameter C, the growth of technology is approximately linear – it is thus compatible with empirical evidence on technology dynamics in most countries of the world. This property of the model suggests that it can be the basis for empirical analysis, which will be performed in the next subsection.

Fig. 4.5 shows the dependence of the average technology after $18N$ Monte Carlo steps on the system size N, for $C = 1$. Every agent comes up with averagely 18 innovations. The larger the system size, the higher the average technology after $18N$ periods. The system can thus be identified with its ability to absorb new technologies. This ability can hence be termed *human capital*. Therefore, a single agent can be identified with an effective unit of human capital in the research and development sector.

4.2.3 Empirical analysis for the OECD countries

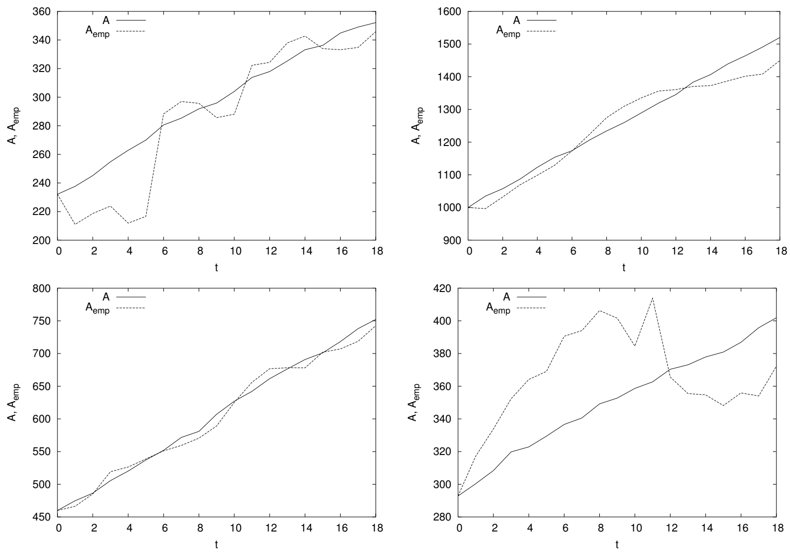
The empirical analysis for the OECD countries was performed in the following way. Again, the period of analysis was 1981-1999. For each country, the size of the system was defined as human capital per researcher in the last period of analysis[7], rounded to the nearest integer. The initial (1981) level of technology was chosen at its empirical level relatively to USA (for USA

[7]The data are given in Tabs. C.5 and C.6 in Appendix C.

the value of 1000 was chosen). The number of Monte Carlo steps was chosen to be the product of the length of the period of analysis (18) and the system size, hence every agent comes up with 18 innovations on average.

Then, the value of the parameter C was chosen in such a way that the model dynamics and empirical dynamics were as consistent as possible. The measure of consistency was the mean relative error (MRE), defined by (3.32).

Figure 4.6: **The empirical and model dynamics of technology in Poland (left, up), USA (right, up), Spain (left, down) and Hungary (right, down)**



The result of the calibration procedure is shown in Fig. 4.6 for Poland, USA, Spain and Hungary. In the case of USA and especially Spain (the best agreement between the model and empirical values) the model and empirical curves are rather consistent. The characteristic property of the model is the emergence of random irregularities in the model curve A, which is otherwise almost linear. For Poland, the agreement between the model and empirical data is not so good, because of a very slow technology growth at the beginning of the analysis period. In the last periods the agreement is, however, very good, which means that the possible reason for the observed slow growth in the early 1980s is just the fact that the empirical data from this period are not very accurate (this was still the period of non-free market economy in Poland). For Hungary, we have the worst result of the calibration procedure. However, we observe such problems with Hungary in all models (compare with e.g. Fig. 3.20), because of anomalous empirical dynamics of technology for this country. For most countries not shown on the plots, the mean relative error is rather small (typically from ca. 1.5% to ca. 3.5%), similar to USA or Spain.

The calibrated values of the parameter C, the system size N, mean absolute error of calibration MRE and the ratio of the maximal to the actual number of imitations are shown in Tab. 4.1. For 19 countries the calibrated value of C ranges from 0.9 to 0.95 (the average value is 0.906). The number of imitations varies from ca. 1% to ca. 2.5% of its maximal number for these countries. In this way, the imitation process takes place after an accumulation of a few innovations by a single agent, such that the difference in the level of technology between an agent and its neighbour exceeds the value of C. The minimal value of the parameter C is 0.78 (for Italy[8]), which means that almost 10% of the maximal number of imitations takes place – the inner technology diffusion mechanism is thus rather effective. Other countries in which this mechanism is effective are: the Czech Republic, Ireland, Mexico, Portugal and Turkey. The only countries for which the calibrated value of C exceeds 0.95 are Poland ($C = 1.00$) and Hungary ($C = 1.12$). Thus, these are countries with low efficiency of inner technology transfer. Apart from Poland and Hungary, these are: Finland, New Zealand and Sweden (the maximal number of iterations in all these countries is more than 100 times the actual value, only in Poland the actual number of imitations is slightly above 1% because of a relatively small human capital stock, which leads to a small value of N).

To sum it up, the simple simulational model of technology diffusion works quite well for most of the countries. Taking different stocks of human capital in different countries (different system sizes), we obtain very close values of

Table 4.1: **The calibrated values of the parameter C, the system size N, mean absolute error of calibration MRE and the ratio of the maximal to the actual number of imitations**

Country	N	C	MRE (%)	% of imitations
AUS	844	0.93	1.72	1.10
AUT	409	0.91	1.78	2.55
BEL	791	0.91	2.28	1.59
CAN	828	0.93	3.19	1.25
CZE	298	0.83	6.42	5.79
DEN	780	0.92	3.21	1.42
FIN	1769	0.95	3.76	0.61
FRA	542	0.89	1.69	2.26
GER	764	0.92	2.21	1.41
GRE	333	0.90	1.80	2.55
HUN	267	1.12	9.93	0.85
IRL	495	0.81	6.77	5.82
ISL	1015	0.91	3.40	1.63
ITA	221	0.78	1.79	9.57
JPN	970	0.91	2.43	1.41
KOR	572	0.90	5.94	2.08
MEX	41	0.79	5.76	10.8
NED	599	0.91	3.58	1.81
NZL	735	0.94	2.55	0.93
NOR	1022	0.91	5.11	1.33
POL	243	1.00	6.20	1.21
POR	163	0.81	4.40	7.34
SPA	287	0.87	1.43	3.77
SWE	1317	0.95	1.87	0.61
SWI	631	0.91	2.56	1.81
TUR	50	0.93	2.39	4.85
UK	597	0.91	2.19	1.86
USA	1123	0.92	2.75	1.36

[8]It confirms the conclusions from other models that technology diffusion is very effective in Italy.

the parameter C for most of the countries. Thus, the mechanism of technological progress can be approximated by single agents' innovations (which are their autonomous inventions or result from technology diffusion from outside) and then imitations by other agents. There is a cost C corresponding to every imitation, which can be regarded as the inventor's premium for his innovation. Such interpretation is also justified by the fact that the calibrated values of C are so close in different countries – if the value of C in some country was much lower than in other countries, then we could observe that the agents (human capital) flow from this country to other countries or to non-research activity. In both cases, the inventor's premium (the value of C) would grow.

The model under analysis can also help us answer the question how to accelerate technological progress – one should increase the premium that innovators get for their inventions. This increase should come from public money, such that the parameter C (the imitation cost of other agents) does not grow. In such case, more human capital would move to research (there would be more research agents) and thus C could go down. The final result would be that more human capital would do research and thus the rate of technological progress would indeed go up, since lower value of C would mean more imitations. Thus, this model justifies public or private grants for research, since such grants increase research intensity over the intensity that results from market mechanisms.

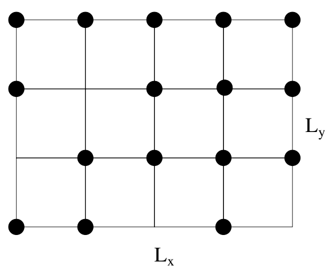
4.3 Microeconomic evolution model with technology diffusion

In this subsection we will analyze a more complex model of technological progress with technology diffusion. This model is a reformulation of the model introduced in [7], where economic agents in an external field were considered and these agents evolved to adjust their characteristics to the value of the field.

4.3.1 Setup of the model

Let us consider a two-dimensional square lattice of $L_x \times L_y$ sites. Each site can be occupied by 0 or 1 agent (firm). Let us assume that the initial concentration of firms equals $c_0 = N(0)/L_xL_y$, where $N(t)$ denotes the

Figure 4.7: **An example of the initial distribution of firms ($N(0) = 16$, $c_0 = 0.8$)**



number of firms at time t. An example of the initial firm distribution in the system (which we will call *country*) is shown in Fig. 4.7.

4

We assume that the i-th firm is characterized by two variables:

- $A_i(t)$ – technology level at time t,
- $w_i(t)$ – the weight of the firm at time t, i.e. its relative market share (for every t, we must thus have $\sum_j w_j(t) = 1$).

Let us assume that the initial distribution of technology levels is uniform on the interval $(0, A_{max})$. For simplicity, we set $A_{max} = 1$. We assume that initially all firms have the same market share $w_i = 1/N(0)$. Then, the initial average technology level, weighted with market shares (such weighted averages will be denoted by $\langle A(0)\rangle$) is equal approximately to half of the maximal level, i.e. $\langle A(0)\rangle \approx 0.5$.

At any time, a firm can:

- go bankrupt – they disappear from the market,
- move through the lattice and look for a partner for collaboration,
- merge with other firms,
- create spin-offs.

We assume that the probability that a firm survives depends on the difference between the firm's level of technology and the frontier technology $F(t)$, the dynamics of which is given by:

$$F(t) = e^{\sigma t}, \tag{4.4}$$

where σ denotes an exogenous parameter, which measures the world technological progress rate (or the progress in the technology leading country). This opens up the possibility for technological diffusion from outside and we will call it the *outer* technology diffusion. Let us also introduce the average relative technology level with respect to the technological frontier: $\langle \tilde{A}(t)\rangle \equiv \langle A(t)\rangle/F(t)$. The system's sensitivity for technological backwardness is expressed by the value of the parameter s, i.e. higher values of this parameter mean that the survival probability is lower for firms whose technology level is off the frontier value. If $s = 0$, in turn, then the system is insensitive with respect to technological backwardness and firms never go bankrupt.

Moreover, we assume that there is a lower bound on the number of firms in the system – we assume that a decreasing number of competitors makes it easier for the surviving firms to stay on the market and the number of firms never goes below N_{min} and the concentration of the firms is always greater or equal c_{min}.

We will now specify the Monte Carlo algorithm for arbitrary time t.

1. We randomly choose a firm indexed by i from the set of $N(t)$ firms.
2. We calculate the probability of survival:

$$p_i = \begin{cases} e^{-s(\langle A(t)\rangle F(t) - A_i(t))} & \text{if } \langle A(t)\rangle F(t) > A_i(t),\ \langle A(t)\rangle < 1 \\ e^{-s(F(t) - A_i(t))} & \text{if } F(t) > A_i(t),\ \langle A(t)\rangle \geq 1 \\ 1 & \text{if } A_i(t) \geq \langle A(t)\rangle F(t),\ \langle A(t)\rangle < 1 \\ 1 & \text{if } A_i(t) \geq F(t),\ \langle A(t)\rangle \geq 1 \end{cases} \tag{4.5}$$

We distinguish two phases, with different influence of the frontier technology:

- phase when the average technology has not yet reached the level $F(0) = 1$ – the first and third formula of eq. (4.5) imply that a firm is certain to survive if their level of technology is equal to at least $\langle A(t)\rangle F(t)$; when it is lower than this value, a non-zero probability of bankruptcy occurs,
- phase when the average technology in the system has reached $F(0) = 1$ – the second and fourth formula of eq. (4.5) imply that a firm is certain to survive if their level of technology is not worse than the frontier technology $F(t)$; when it is lower than $F(t)$, there is a non-zero probability of bankruptcy.

The existence of the first phase is thus an additional advantage of the countries in which the technology is relatively bad and makes possible a faster growth by making sure that the best firms are certain to survive, even if their technology level is still worse than the initial level of the frontier technology.

3. We draw a number r from the uniform distribution.
   - If $r > p_i$, the firm goes bankrupt and the lattice site becomes empty. The weights of other firms grow proportionally, such that the normalization condition $\sum_i w_i(t) = 1$ holds. We return to point 1 of the algorithm.
   - If $r \leq p_i$, the firm survives and tries to move to a neighbouring lattice site. We draw a number r_1 and if $r_1 < 0.25$, we check whether the site which is north of the firm's site is empty, if $0.25 \leq r_1 < 0.5$ we check the site which is west etc.
4. If the neighbouring site is empty, the firm moves to this site and checks whether there is some other firm in the nearest neighbourhood of the new site.

   If such firm is absent, the firm profits from *outer* technology diffusion, according to the formula:

$$A_i(t) \rightarrow A_i(t) + r_2(F(t) - A_i(t)), \tag{4.6}$$

   where r_2 is a number drawn from the uniform distribution. Then we come back to point 1 of the algorithm. In this way, the *outer* technology diffusion means the imitation of the frontier technology. However, usually it is not possible to introduce the frontier technology fully and the level of technology of the firm grows by a fraction r_2 of the difference of the frontier technology and the current technology of the firm.
5. If there is some firm j in the neighbourhood of the firm i, then:
   - with probability b (which is a parameter of the model), the firms merge. The technology of the new firm is:

$$\begin{aligned} &\text{VAR. 1:} \quad A_i(t) \rightarrow 0.5(A_i(t) + A_j(t) + 0.5r_3|A_i(t) - A_j(t)|), \\ &\text{VAR. 2:} \quad A_i(t) \rightarrow \max\{A_i(t), A_j(t)\}, \end{aligned} \tag{4.7}$$

where r_3 is a number drawn from the uniform distribution. The firm j disappears from the system and the weight of the new firm is equal to the sum of weights of the merging firms. In the first variant, the technology of the new firm is the arithmetic mean of the technology levels of the merging firms plus some synergy effect, which is larger if the difference in technologies is larger. In the second variant, the new firm's level of technology is equal to the technology level of the technologically more advanced firm The synergy effects will be called the *inner* technology diffusion.

- With probability $1-b$, the firms i and j create a spin-off. A firm k emerges and it is located in the 8-site neighbourhood of the firm i (north, north-west, west, south-west etc.). The positioning procedure is analogous to the one in point 3 (a number r_4 is drawn and depending on the outcome a suitable site is chosen). If the appropriate site is not empty, the spin-off does not emerge. The technology of the spin-off equals (there are again two variants, analogous to the case when firms merge):

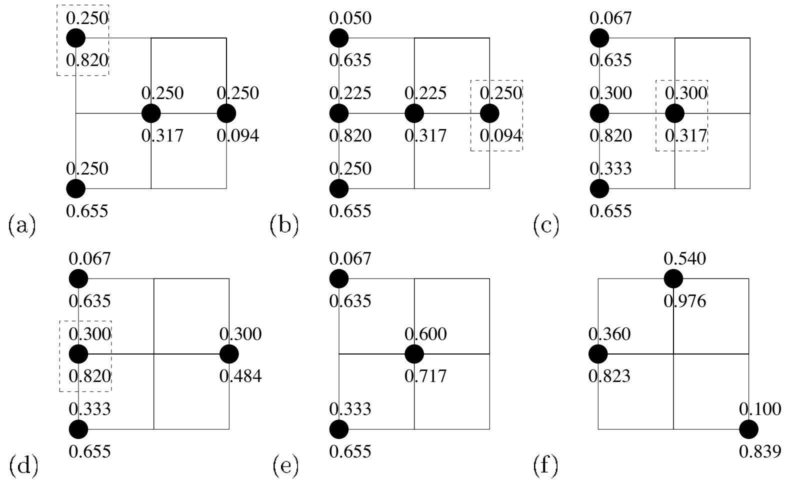
$$\begin{aligned} &\text{VAR. 1:} \quad A_i(t) \to 0.5(A_i(t) + A_j(t) + 0.5r_5|A_i(t) - A_j(t)|), \\ &\text{VAR. 2:} \quad A_i(t) \to \max\{A_i(t), A_j(t)\}, \end{aligned} \tag{4.8}$$

where r_5 is a number drawn from the uniform distribution. Hence, the economy profits from the *inner* technology diffusion also if a spin-off is created. The weight of the new firm is equal to the sum of weights of firms i and j, multiplied by a parameter $w_s \in [0,1]$. The extreme values of the parameter w_s correspond to the cases that the spin-off can never emerge ($w_s = 0$) or if the firms i and j disappear when the spin-off emerges ($w_s = 1$). The weights of firms i and j decrease accordingly, by $w_i w_s$ and $w_j w_s$.

6. We return to point 1 of the algorithm until $N(t)$ firms have been chosen. Then, a Monte Carlo step is finished, i.e. we set $t \to t+1$. The random choice of a firm in point 1 of the algorithm implies that a given firm can be chosen more than once at time t and hence firms which are not chosen at this Monte Carlo step exist.

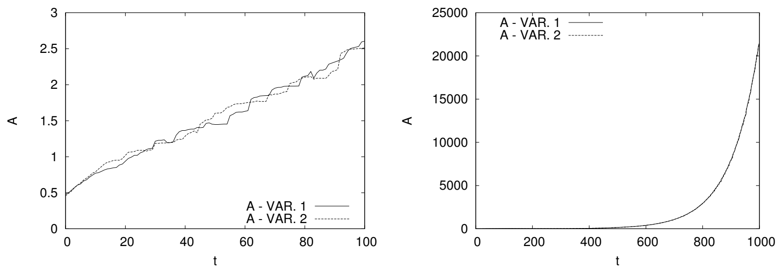
We will now present an example of the application of this algorithm to a simple 3×3 system with initially four firms – Fig. 4.8. Fig. (a) shows that the firm (1,1) (we number the sites from the upper left corner) was chosen. The probability of survival of this firm equals 1, since the level of technology of this firm exceeds $\langle A(0)\rangle = 0.472$. The firm moves to site (2,1) ($r_1 = 0.726$) and creates a spin-off with firm (2,2). The spin-off emerges at site (1,1). The weight of the spin-off is 0.05 and the level of technology is larger than the arithmetic mean of the parent-firms – thus the average technology in the system grows to 0.475 (Fig. (b)). Hence, we observe *inner* technology diffusion. Next, the firm (2,3) is chosen. Its probability of survival is 0.683. We draw $r = 0.701$ and the firm (2,3) goes bankrupt. The (weighted) average level of technology of the remaining firms is 0.602. On Fig. (c), the firm (2,2) is chosen. Its survival probability equals 0.752. We obtain $r = 0.123$ (the firm survives) and moves to site (2,3) ($r_1 = 0.916$). The lack of neighbours means that we observe the *outer* technology diffusion and the level of technology of this firm goes up from 0.317 to 0.484; the average technology $\langle A(0)\rangle$ rises to 0.652. On Fig. (d) the firm (2,1) is chosen. The survival probability is 1 and the firm moves to site (2,2) and merges with firm (2,3). The new firm's technology is the arithmetic mean of the parent-firms plus some synergy effect (*inner* technology diffusion). We obtain the

Figure 4.8: **An example of the evolution of a 3×3 system**



Explanation: parameter values: $\sigma = 0.01$, $s = 1$, $b = 0.1$, $w_s = 0.1$. The upper number denotes the weight, the lower number the technology level.

Figure 4.9: **Average technology dynamics in the mid- and long-term**



situation depicted in Fig. (e). The zeroth Monte Carlo step is finished (the algorithm involved $N(0) = 4$ firms) and the average level of technology is now 0.691. Fig. (f) shows an example of the situation after 10 Monte Carlo steps. At this time, there are 3 firms in the system and the average technology level equals $\langle A(10) \rangle = 0.907$. However, the technology frontier moved to $F(10) = 1.105$, so the average relative technology is now $\langle \tilde{A}(10) \rangle = 0.821$.

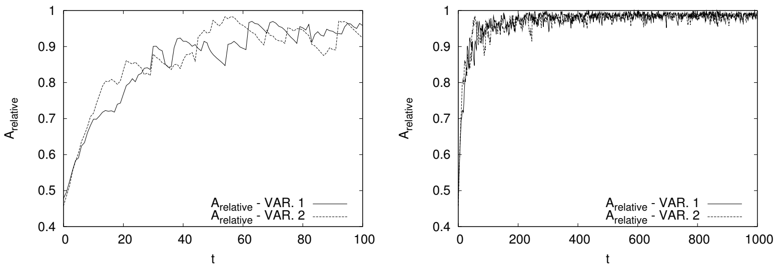
4.3.2 The properties of the model

We will now examine the properties of the model. We show the simulation results for a 10×10 system, with an initial concentration of firms equal to 0.8 and the following parameters: $\sigma = 0.01$, $s = 1$, $b = 0.01$, $N_{min} = 10$, $w_s = 0.1$.

Fig. 4.9 depicts the dynamics of the average technology (denoted A in the figure) in the mid- and long-term. Mid-term means here that the system has not yet reached the frontier technology level (according to Fig. 4.10, which shows the average relative technology (denoted $A_{relative}$), such level is reached after ca. 100 Monte Carlo steps). This period is characterized by an approximately linear technology growth (in both variants). Such growth in technology results from the dominating mechanism of technological progress – the *inner* technology diffusion, i.e. synergy effects when the firms merge or cooperate to create spin-offs and also the bankruptcies of the firms with lowest technology levels.

In the long-term, however, the *outer* technology diffusion dominates, which results in an exponential growth of technology with the rate σ (with

Figure 4.10: **Average relative technology dynamics in the mid- and long-term**



small fluctuations around the frontier technology – Fig. 4.10). When the average relative technology level is at a given time t close to $F(t)$, then the further growth in technology can be interpreted as autonomous technological improvements, and not outer technology diffusion.

Fig. 4.11 shows the dynamics of the number of firms (N) in the mid- and long-term. This suggests that we can distinguish three phases in the system behaviour:

- first phase – the firms with the worst technology disappear, approximately linear growth of average technology, fast growth in average relative technology,
- second phase – stabilization of the number of firms at ca. 50% of the initial number (since around 50% of the initial number of firms has technology worse than average; in the second variant the number of firms is a little larger, since when the firms merge, the new firm has better technology than in variant 1), approximately linear growth in average technology (at a rate similar to phase 1) and a much slower growth in average relative technology (since the firms are closing in on the technological frontier),
- third phase – after the firms reach an average level of technology which is close to the frontier technology level, the number of firms starts to decrease, until it reaches the minimum number – such behaviour maximizes the growth rate of technology (the rate σ is reached), which is impossible to achieve with too much competition on the market.

Figure 4.11: **The dynamics of the number of firms in the mid- and long-term**

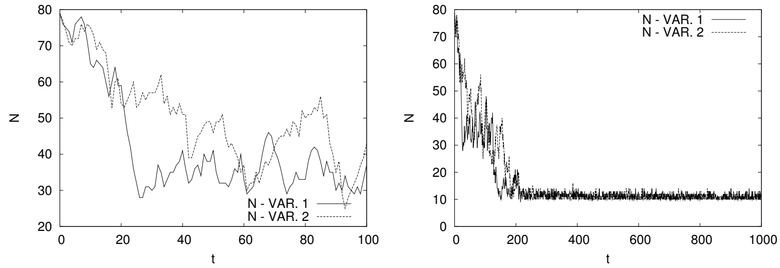
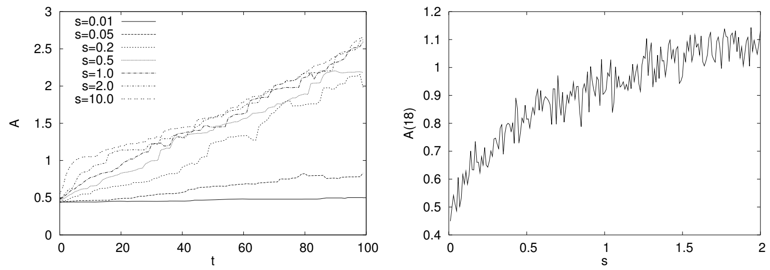


Figure 4.12: **The dynamics of the average level of technology (left) and the average level of technology at $t = 18$ (right) for selected values of s**



The sensitivity of the system to the value of the parameter s was also investigated. In the left of Figs. 4.12-4.14 one can see the dynamics of the average level of technology, average level of the relative technology and the number of firms for selected values of the parameter s, in the right the same quantities at time $t = 18$ for different s.

The higher the value of s, the faster the growth of the average level of technology and the average level of relative technology. For s much more than 1, after around 10-15 full Monte Carlo steps the mechanisms of *outer* technology diffusion and autonomous technological improvements start to dominate and the rate of growth is similar and independent of s. In the right sides of Figs. 4.12 and 4.13 we can then observe that for values of s more than ca. 1.5 the technology level at time $t = 18$ does not depend on the value of s and the average level of relative technology is close to 1.

Figure 4.13: **The dynamics of the average level of relative technology (left) and the average relative level of technology at $t = 18$ (right) for selected values of s**

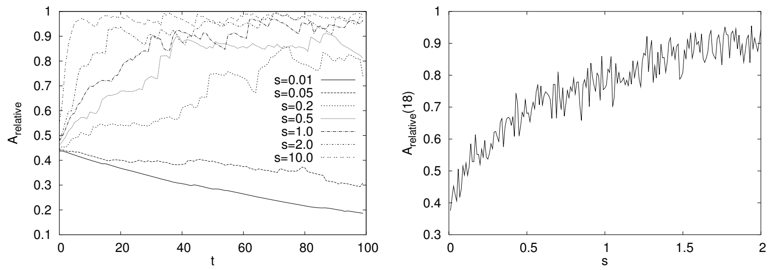
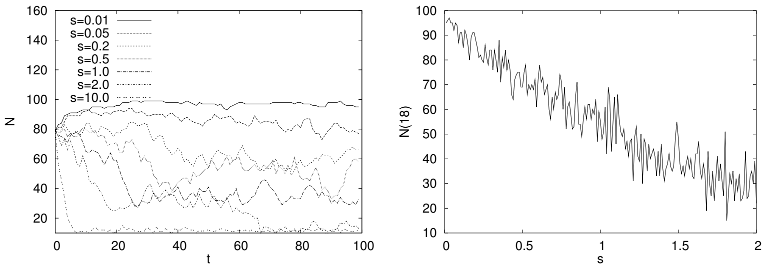


Figure 4.14: **The dynamics of the number of firms (left) and the number of firms at $t = 18$ (right) for selected values of s**



We also observe an inverse dependence between the number of firms and the value of the parameter s (Fig. 4.14). If the sensitivity to technological backwardness is low (small s), the number of firms is close to maximal. The than 1, after around 10-15 full Monte Carlo steps the mechanisms of *outer* technology diffusion and autonomous technological improvements start to dominate and the rate of growth is similar and independent of s. In the right sides of Figs. 4.12 and 4.13 we can then observe that for values of s more than ca. 1.5 the technology level at time $t = 18$ does not depend on the value of s and the average level of relative technology is close to 1.

We also observe an inverse dependence between the number of firms and the value of the parameter s (Fig. 4.14). If the sensitivity to technological backwardness is low (small s), the number of firms is close to maximal. The higher the value of s, the smaller the number of firms in the system, on average. For large values of s a dozen or so number of periods is enough for the number of firms to decrease to a value close to N_{min}. The decrease in the number of firms at $t = 18$ with respect to the value of s is approximately linear. For the values of $s < 2$, the minimal number of firms is not reached and thus both *inner* and *outer* technology diffusion takes place in the model in the analyzed time horizon.

4.3.3 Empirical analysis for the OECD countries

Empirical analysis for the OECD countries was performed in the following way. The period of analysis was again chosen to cover the years 1981-1999. The system size was assumed to be 10×10 and the initial concentration $c_0 = 0.8$. The minimal number of firms in each country is $N_{min} = 10$ and the parameter b was chosen at 0.01.

The initial (1981) technology level for the i-th country was assumed to be at its empirical level $A_i^{emp}(0)$ (relatively to USA). To achieve this, the economic agents in the systems were assigned technology levels A from the uniform distribution on the interval $(0, 2A_i^{emp}(0))$, if $A_i^{emp}(0) \leq 0.5$, or on the interval $(1 - 2A_i^{emp}(0), 1)$, if $A_i^{emp}(0) > 0.5$ (in this way, no agent had an initial technology of more than 1).

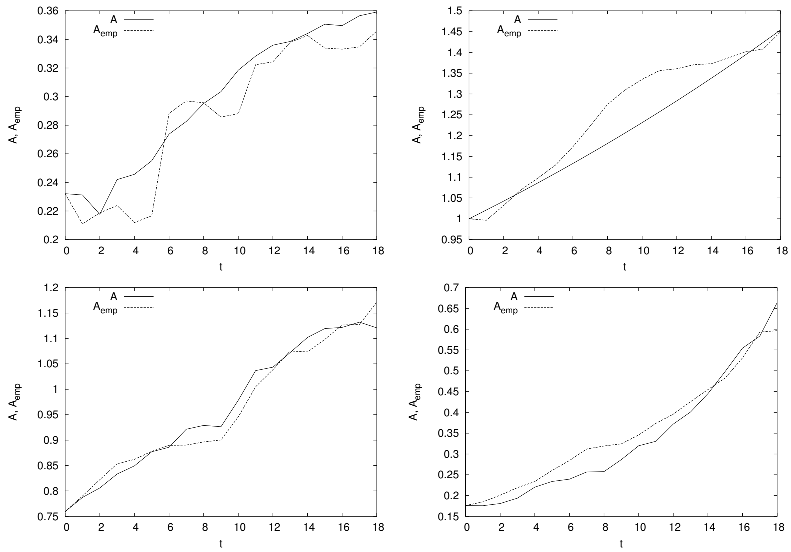
The technology level for USA was assumed to be the frontier technology and it was assumed that it grew at the average empirical growth rate of (for 1981-1999) 2.08%. Hence, the technology frontier function is given by $F(t) = e^{0.0208t}$.

Then, the value of the parameter s was chosen in such a way that the mean relative error (defined by eq. (3.32)) that measures the agreement between the model and empirical dynamics took the smallest possible value.

The results of the calibration procedure for the parameter s are shown in Fig. 4.15 for Poland, Belgium and South Korea. The dynamics of the frontier technology is also shown, together with the empirical curve for USA.

The case of Belgium is shown to illustrate the best agreement between the model and empirical data. An important property of the model curve is the presence of random irregularities, similar to empirical fluctuations. Such irregularities occur for all the countries and since their scale is of the same order as in empirical data, we can suppose that the empirical fluctuations are also random and do not reflect any particular mechanism (such as some forms of cyclical behaviour). If we accept this conclusion, the agreement of the model and empirical data is rather good for most of the countries.

Figure 4.15: **Empirical and model technology dynamics for Poland (left, up), USA (right, up), Belgium (left, down) and South Korea (right, down)**



However, for Poland this agreement is not very good and for South Korea the worst for all of the countries, but still it is not very bad. The mean relative error varies from ca. 1.9% to ca. 10% (Tab. 4.2).

The results of the calibration procedure of the parameter s are shown in Tab. 4.2. We observe very different values of this parameter for different countries – from very small (0.02 for Hungary – this country however has very strange empirical dynamics and thus we can not conclude much about it) to very high (1.98 for Switzerland). In this way, we can distinguish three groups of countries:

- countries in which the *inner* technology diffusion dominates (low values of s),
- countries in which the *inner* and *outer* technology diffusion play a similar role ($s \approx 1$),
- countries in which the *outer* technology diffusion or autonomous innovations dominate (high values of s).

Table 4.2: **The results of the calibration of the parameter s and the corresponding mean relative errors**

C'try	s	MRE	C'try	s	MRE	C'try	s	MRE
AUS	0.80	2.17	GRE	0.58	2.77	NZL	0.46	2.63
AUT	0.70	3.39	HUN	0.02	8.21	NOR	1.73	4.05
BEL	1.46	1.86	IRL	1.38	4.22	POL	0.73	5.37
CAN	1.87	6.86	ISL	1.29	3.29	POR	0.74	5.61
CZE	0.55	5.77	ITA	1.66	2.53	SPA	0.32	3.69
DEN	1.34	3.46	JPN	0.48	2.25	SWE	1.15	2.73
FIN	0.97	2.81	KOR	1.38	9.95	SWI	1.98	2.65
FRA	1.69	3.04	MEX	0.39	6.32	TUR	0.36	2.82
GER	0.94	2.84	NED	0.83	2.04	UK	1.36	3.28

Explanation: MREs given in percent values.

The first group of countries consists of Austria, the Czech Republic, Greece, Spain, Japan, Mexico, New Zealand, Poland, Portugal, Hungary and Turkey[9]. This group of countries is diversified, but it consists mainly of the developing countries, in which the technological advancement is not the most important factor that determines the probability of survival of firms (the sensitivity to technological backwardness is small). The presence of such countries as Japan means that low sensitivity to technological backwardness can also be the property of the most developed countries.

The third group of countries consists mainly of highly-developed countries (with the exception of South Korea – a developing country in the period of interest). In such countries, with long traditions of free market economy, the level of technology is one of the most important factors of competition between companies – thus the underdeveloped firms do not survive for long.

The second group of countries consists of a relatively small number of countries (Austria, Finland, Germany, The Netherlands, Sweden) and it can be interpreted as interpolating between the two other groups. The case of Germany, a country which was divided into two independent states for half of the analyzed period, seems to correspond well with this interpretation.

The general conclusion that can be drawn from this model is that the sensitivity to technological backwardness is rather small in the developing countries and rather high in the highly-developed states. Thus, it is probable that the mechanisms of technological progress are quite different in these groups of countries. In the developing countries, technological progress consists mainly in technology transfers from highly-developed companies (which can be interpreted as e.g. firms with international capital) to the underdeveloped ones. In the highly-developed countries, in turn, companies' technology levels are more close to one another and the dominating mechanism of progress is the development of autonomous innovations or the use of the most developed technologies available in the world.

4.4 Summary

In this chapter, we analyzed an alternative approach to the modelling of technological progress. We argued that the standard neoclassical approach

[9]The introduced classification of countries is not very strict. Here we assume that low value of s is less than 0.8, $s \approx 1$ means $s \in [0.8, 1.2]$ and $s > 1.2$ is a high value of this parameter.

4

of *representative agents* is a severe limitation from the point of view of modelling. In particular, it does not allow to analyze the effects of heterogeneity of agents, which is an obvious property of the real world. We showed that a consequence of this heterogeneity can be fluctuations in the growth rates of technology. These are also observed in the real world. The class of simulational models makes it possible to endogenize technology diffusion between the countries and also between domestic companies.

We considered two models. In the first, the growth in technology results from autonomous innovations of agents, which can also result from technology diffusion from other countries. Then, the innovations can be imitated by other agents, who buy the new technology (patent) if it is profitable for them. We showed that such approach leads to interesting empirical conclusions.

The second model is more complex – the agents can move on a lattice, merge, cooperate, go bankrupt and use external technology (from the technology leading country). The empirical conclusions from this model are also rather interesting.

Chapter 5

Simulational models of simple economies with human capital and technological progress

In this chapter we will introduce two simulational model of simple economies. In the first, economic growth results from the accumulation of three types of capital – *physical*, *human* and *technological* capital. In this way, we will show that for some parameter values simulational models (with *heterogeneous agents*) reduce to the well-known neoclassical models (with *representative agents*). For other parameter values, however, simulational models can describe random and cyclical fluctuations in production and investments, which happen in the real world, but are absent in neoclassical models. In the second model, we will introduce an endogenous mechanism of technological progress and human capital accumulation.

5.1 Solow-Swan-type simulational model

5.1.1 Setup of the model

Let us consider an economy consisting of N firms (sectors)[1]. The factors that enter the production function are: physical capital, human capital and technology. By *technology*, we understand the accumulated R&D expenditures, multiplied by an efficiency factor. Therefore, we can speak of three types of capital: physical, human and technological capital. This justifies the use of the term *Solow-Swan-type model*, since the growth mechanism for these three types of capital is similar and analogous to the growth mechanism in the Solow-Swan model – where physical capital accumulation plays the key role. The main difference with respect to this model (and other models analyzed within the framework of optimal control theory) is the introduction of *heterogeneous* (*individualized*) agents, in the place of *representative* agents. Formally, this is achieved by setting different parameter values for different agents (firms), taking these values from some probability distribution.

The production of each firm (sector) is given by the multiplicative function:

$$F(A, K, H) = A^{\theta_A} K^{\theta_K} H^{\theta_H} \tag{5.1}$$

(where A denotes technology, K – physical capital, H – human capital, θ_A, θ_K and θ_H are parameters) and it is divided in the following way.

- The fraction c, where c is a normal random variable[2] is consumed. The variable c can also be interpreted as the production cost (in this way,

[1]Different firms have different parameter values. If one assumes that a few firms in one sector have the same (or very close) parameter values, one can interpret this basic entity of the model as a sector and speak of N sectors in the economy.

[2]Here, a normal distribution is by default a normal distribution confined to some interval of economic interest. For example, the variable c can have values in [0,1].

physical and human capital are accumulated only from profits) or as the return to the factors of production,

- The remaining fraction is divided into div parts between the investments in technology I_A, human capital I_H and physical capital I_K in such a way that each project is associated with the currently (after allocating previous parts accordingly) most effective investment. The size of each "project" is equal to the firm's profit, divided by div and multiplied by a normal variable $size$ with an expected value of 1 and a standard deviation of 0.1. This means that the projects differ in size to some extent[3]. The efficiency of an investment is calculated as:

$$e_x = \hat{\alpha}_x \frac{\partial F(A,K,H)}{\partial x}, \tag{5.2}$$

  where $\hat{\alpha}_x$ denotes the expected value of the conversion (efficiency) coefficient per unit of x. The conversion coefficients $\alpha_x(t)$ are normal random variables.

The conversion coefficients are chosen to yield the balanced growth of all three types of capital. This arbitrary choice is a simplification of the model.

Technology, physical capital and human capital depreciate with rates δ_A, δ_K and δ_H, which are normal random variables. The dynamics equations of the model are:

$$A(t+1) = (1-\delta_A(t))A(t) + \alpha_A(t)I_A(t), \tag{5.3}$$

$$K(t+1) = (1-\delta_K(t))K(t) + \alpha_K(t)I_K(t), \tag{5.4}$$

$$H(t+1) = (1-\delta_H(t))H(t) + \alpha_H(t)I_H(t), \tag{5.5}$$

In a single Monte Carlo step (time t) this procedure (production + depreciation + division of product between consumption and investments) encompasses all firms. At each step, we can calculate the aggregated quantities for the whole economy: technology, physical capital, human capital, production, consumption etc. This makes possible to determine the growth paths of these quantities for the economy under consideration.

Table 5.1: **An example of division of profit 0.2 between investments in 3 types of capital**

τ	left	A	K	H	Y'	$size$	e_A	e_K	e_H	i	αI
1	0.200	1.000	0.940	0.940	0.940	0.022	0.313	0.333	0.333	H	0.011
2	0.178	1.000	0.940	0.951	0.947	0.020	0.316	0.336	0.332	K	0.022
3	0.158	1.000	0.962	0.951	0.955	0.021	0.318	0.331	0.335	H	0.010
4	0.137	1.000	0.962	0.961	0.961	0.017	0.320	0.333	0.333	H	0.008
5	0.120	1.000	0.962	0.969	0.967	0.022	0.322	0.335	0.333	K	0.024
6	0.098	1.000	0.986	0.969	0.975	0.019	0.325	0.329	0.335	H	0.010
7	0.079	1.000	0.986	0.979	0.981	0.019	0.327	0.332	0.334	H	0.009
8	0.060	1.000	0.986	0.988	0.987	0.023	0.329	0.334	0.333	K	0.019
9	0.037	1.000	1.005	0.988	0.994	0.022	0.331	0.330	0.335	H	0.013
10	0.015	1.000	1.005	1.001	1.002	0.015	0.334	0.332	0.334	A	0.005
		1.005	1.005	1.001	1.007						

Explanation: We assume the production function $F(A,K,H) = AK^{1/3}H^{2/3}$. Initially, $K = H = A = 1$. The columns are: τ – step, left – profit to be divided, A, K, H – the size of the relevant capital after successive projects, after depreciation (6% for K and H, no depreciation of technology), Y' – production from given A, K, H, $size$ – size of the project, e_x – current efficiency of investment in x, i – the currently most efficient factor, $\alpha_x I$ – growth of the relevant capital after new project. The conversion coefficients are: $\alpha_A = 1/3$, $\alpha_K = 1$, $\alpha_H = 1/2$.

[3] The last project (the div-th one) uses all of the undivided profit after $div - 1$ divisions.

Tab. 5.1 shows an example of the division of profit between investments in technology, physical capital and human capital, for initial production of

Table 5.2: **Means and standard deviations of random variables**

	c	δ_K	δ_H	δ_A	α_A	α_K	α_H	$size$	div
mean	0.8	0.06	0.06	0.0	0.33	1.00	0.50	1.0	10
std.dev.	0.08	0.01	0.01	0.0	0.03	0.10	0.05	0.1	0

1 and the consumed fraction of 80%, 20% allocated to $div = 10$ investment projects. Since there is no depreciation of technology, restitution investments in physical and human capital dominate – only after full restitution of these capitals an investment in technology is undertaken. After all projects are completed, we observe production growth from 1 to 1.007. The production 1.007 is then divided between consumption and investments with a similar algorithm.

5.1.2 Properties of the model

To examine the properties of the model, we use the parameter values given in Tab. 5.2 and the production function:

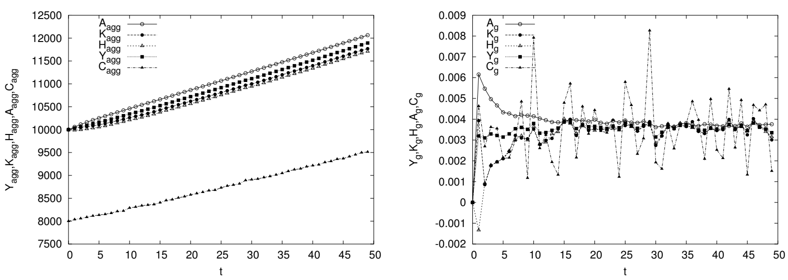
$$F(A, K, H) = A^{1/2} K^{1/6} H^{1/3}. \tag{5.6}$$

Such form of the production function was chosen to eliminate increasing returns to scale that emerge when the sum of exponents exceeds 1.

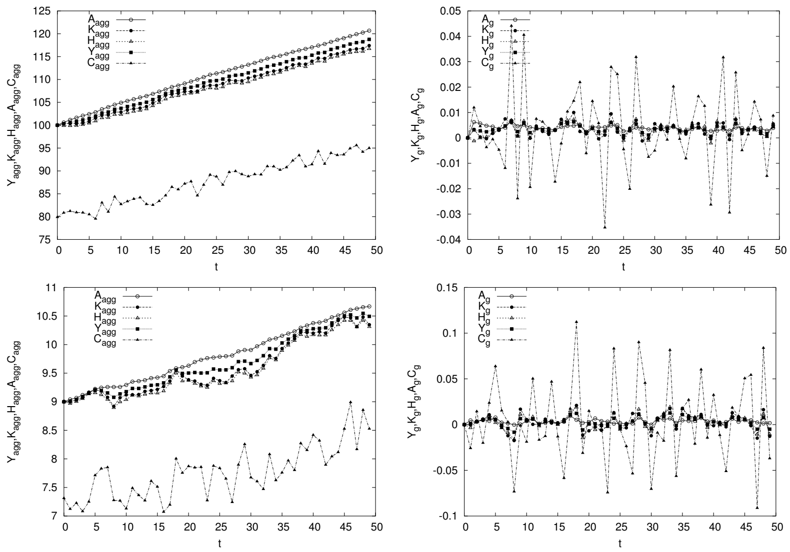
Fig. 5.1 shows the influence of the number of firms (sectors) on the dynamics of production. In the case of a large number of firms, the trajectories are *smooth* and the *fluctuations are almost absent*. Such behaviour is intuitive – in the limit of the infinite number of firms, the fluctuations cancel out and we obtain the same result as for a set of representative agents, which are analyzed in the neoclassical models of economic growth. The examined model can then be considered as a particular case of the Solow-Swan model with three types of accumulated capital.

When the number of firms decreases, the *fluctuations start to play an increasing role*. In the case of only 9 firms (sectors), we observe the presence of cyclical fluctuations. The largest fluctuations are observed in the dynamics of consumption, for which fluctuations resulting from the variety of firms and fluctuations in the overall consumption rate are superimposed[4].

Figure 5.1: **The influence of the number of firms on production**



[4]We assume that every firm allocates the same fraction of production and this fraction is a normal random variable.



Explanation: left: the paths of 3 types of capital, production and consumption (aggregated for the whole economy – bottom index agg) for, respectively, 10000, 100 and 9 firms. Right: the growth paths (bottom index g) of the respective quantities.

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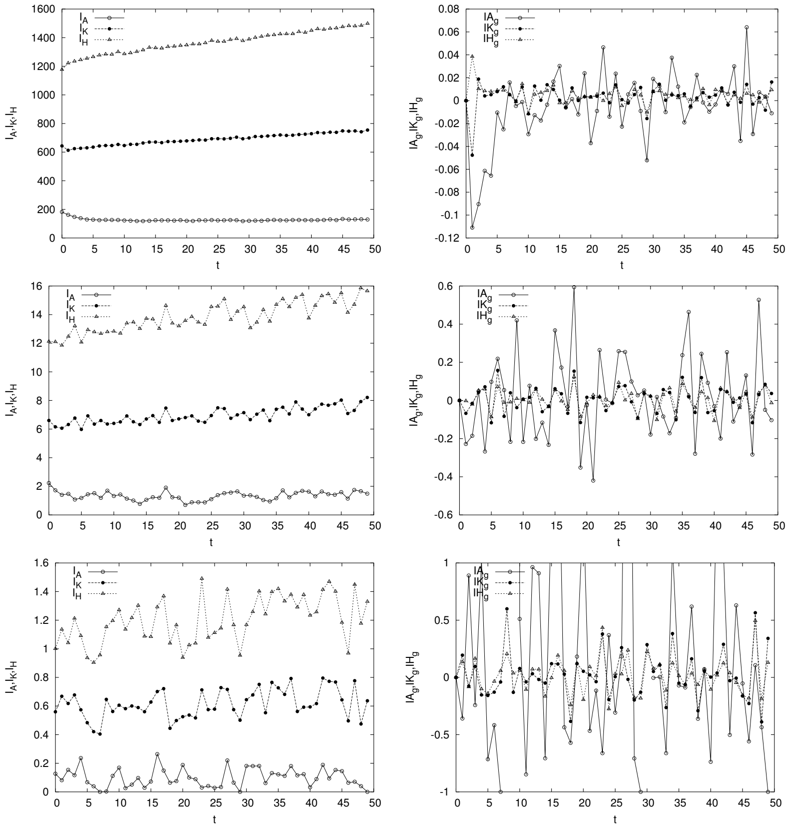
The behaviour of the growth rates of the considered quantities is similar. For large number of firms we obtain small fluctuations (except for consumption fluctuations, independent of the variety of firms) and we can speak of a balanced growth of production and the three types of capital. The smaller the number of firms, the more visible the fluctuations (the vertical scale is different for these plots) – they are of the order of, respectively, 0.1%, 0.5% and 1.5%).

Fig. 5.2 shows the influence of the number of firms (sectors) on the dynamics of investments in three types of capital. The behaviour of these quantities explains the presence of fluctuations of production and the three types of capital, which are observed in Fig. 5.1. In the case of a large number of firms, the fluctuations are almost absent – averaging the investments over a large number of firms leads to results that are similar to the ones from the neoclassical models with representative agents. The smaller the number of firms, the larger the investment fluctuations. We observe a kind of cyclical behaviour that leads to a cyclical behaviour also for production and the three types of capital. The fluctuations in the growth rates of investments are even larger than in the growth rates of production and capitals. For a small number of firms, we observe that there are periods without investments in technology – the growth rate of technology is then -1 and is indefinite in the subsequent period.

Fig. 5.3 shows the influence of the parameter div on the dynamics of production, three types of capital, consumption and investments for $N = 100$ firms. The larger the value of this parameter, the more effective the allocation of investments (in the limit $div \to \infty$ we would have an ideally effective allocation – every infinitesimal fraction of profit would go to the most effective type of capital). At the scale of the whole economy, any potential inefficiency related to the activity of a single firm is equalized by other firms' activities and the influence of the parameter div is practically unobservable.

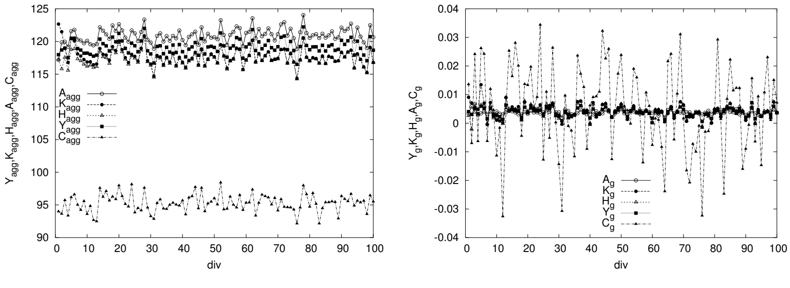
A similar analysis was performed for the parameter c and it is shown in Fig. 5.4. The larger fraction of profit goes to investments (smaller c), the higher the production. However, in practice, too low values of the parameter c are impossible to reach in real-world economies, since the parameter c can be treated as return to the factors of production (or production costs), so the realistic values are above ca. 70%. The value $c \approx 82\%$ is the break-even point for the economy – for higher values of c it is not possible to reproduce the factors of production which are depreciated and production in subsequent periods starts to decrease. Also, the investments in technology disappear,

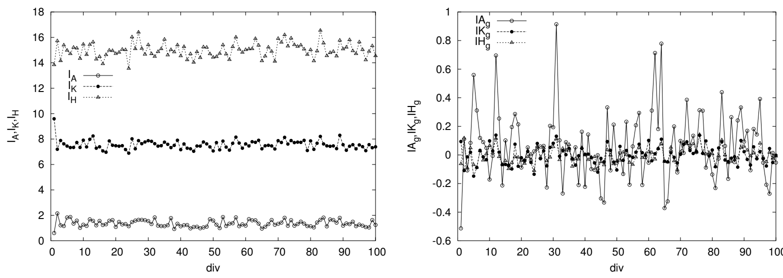
Figure 5.2: **The influence of the number of firms on the dynamics of investments**



Explanation: left: the paths of investments in 3 types of capital (aggregated for the whole economy) for, respectively, 10000, 100 and 9 firms. Right: the growth paths of investments (bottom index g).

Figure 5.3: **The influence of the parameter div on the dynamics of production, 3 types of capital, consumption and investments**





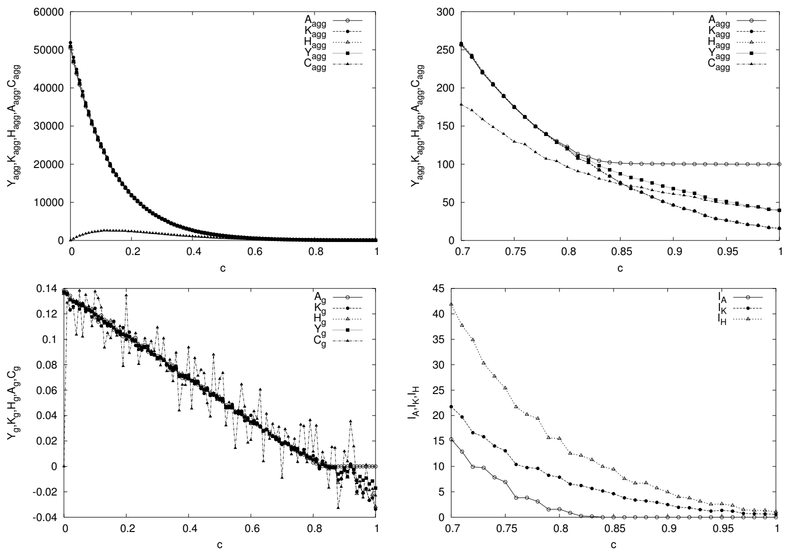
Explanation: left: aggregated quantities, right: respective growth rates. All plots for $t = 50$ and $N = 100$ firms.

since it is more reasonable to invest all the unconsumed profit in physical and human capital, which undergo depreciation. Therefore, the savings rate $1-c$ determines the growth rate of production p.c. and the remaining variables of the model.

5.1.3 Discussion

The model that we considered in this section is too simple to serve as a realistic description of real-world economies and especially the role of human capital and technological progress for economic growth. However, we showed the close relations between neoclassical and simulational models. For small diversity of agents (firms, sectors) or very large number of them, the simulational model reproduces the most important results of the neoclassical

Figure 5.4: **The influence of the parameter c on the dynamics of investments**



Explanation: the standard deviation of c was taken at 10% of its mean value. Up: consumption, three types of capital and production. Down, left: the growth rates of these quantities, right: investments in three types of capital. All plots for $t = 50$ and $N = 100$ firms.

models – the *heterogeneous agents* are almost identical or the diversity is averaged out and we have a situation analogous to a set of *representative agents*.

For larger diversity of agents (or for smaller number of agents, which is equivalent), there emerge fluctuations and cyclical behaviour in the simulational model – these are inherent properties of the real-world economies. This feature of the model makes it possible to describe this very complex system – the economy. Another advantage of the simulational approach is its elasticity – it is possible to choose very different sets of rules for the agents and analyze the consequences. In this way, one can overcome one of the most important constraints of the neoclassical models – their extensions in certain directions (e.g. taking the diffusion of technology effects into account) is often very difficult, since only some formulations of the problem are subject to analytical solution.

An example of the extension of the simulational model to include endogenous mechanisms of human capital accumulation and technological progress is considered in the following section.

5.2 Ramsey-Cass-Koopmans-type simulational model

5.2.1 Setup of the model

Let us consider an economy consisting of N agents (firms, sectors). The agents produce a single kind of good (which can be thought of as an aggregate good which can serve both as a consumer good and an investment good). The production of the firms is given by the following production function:

$$Y_i(t) = (K_i(t))^{\alpha} (A_i(t) n_i(t) H_i(t))^{1-\alpha}, \tag{5.7}$$

where: α – a parameter, the bottom index i denotes i-th firm, t – time, $K_i(t)$ – physical capital, $A_i(t)$ – technology level, $H_i(t)$ – human capital stock, $n_i(t)$ – the fraction of human capital used in production (the remaining part, $1 - n_i(t)$, is used in human capital accumulation).

Each firm (sector) can be identified with a single consumer or a group of consumers who own it. The basic task of the owner(s) is to divide the production in any given period of time between consumption and investments in physical capital, human capital and technology. The aim of the owner(s) at time t is to maximize the utility of consumption over a given time horizon $[t,\, t + T_{max}]$, where T_{max} is an important parameter to be specified later.

At every time step t, we have:

$$Y_i(t) = C_i(t) + I_i^K(t) + I_i^H(t) + I_i^A(t), \tag{5.8}$$

where: $Y_i(t)$ – i-th firm's production at time t, $C_i(t)$ – consumption of the owner(s) of the i-th firm, $I_i^x(t)$ – i-th firm's investment in x at time t ($x = K,\, H,\, A$)[5]. For simplicity, we allow no borrowing, i.e. in each period of time only the produced amount of goods can be divided. We introduce the decision variables[6] $\hat{s}_i^K(t)$, $\hat{s}_i^H(t)$ and $\hat{s}_i^A(t)$, denoting, respectively, the fraction of i-th firm's output allocated to an investment in physical capital, human capital and technology. Thus:

[5] We will call Y, C, K, H, A the *state variables*.

[6] All decision variables that will be control variables of the maximization problem will be denoted by hats above their symbols.

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$$I_i^x(t) = \hat{s}_i^x(t)Y_i(t), \qquad x = K,\, H,\, A, \tag{5.9}$$

$$C_i(t) = (1 - \hat{s}_i^K(t) - \hat{s}_i^H(t) - \hat{s}_i^A(t))Y_i(t). \tag{5.10}$$

In this way, the balance condition (5.8) can be written as:

$$Y_i(t) = (1-\hat{s}_i^K(t)-\hat{s}_i^H(t)-\hat{s}_i^A(t))Y_i(t)+\hat{s}_i^K(t)Y_i(t)+\hat{s}_i^H(t)Y_i(t)+\hat{s}_i^A(t)Y_i(t). \tag{5.11}$$

The dynamics equations for K, H and A are:

$$K_i(t+1) = \left(1 - \delta_i^K(t)\right) K_i(t) + \Delta K_i(t), \tag{5.12}$$

$$H_i(t+1) = \left(1 - \delta_i^H(t)\right) H_i(t) + \Delta H_i(t), \tag{5.13}$$

$$A_i(t+1) = \left(1 - \delta_i^A(t)\right) A_i(t) + \Delta A_i(t), \tag{5.14}$$

where: $\delta_i^x(t)$ – the depreciation rate of variable x_i at time t, $\Delta x_i(t)$ – the increase of x_i as an effect of the respective investment. The functional forms of $\Delta x_i(t)$'s can be chosen in an almost arbitrary way.

So far, our considerations have been fairly general. Now, let us try to be more specific and assume the following dependences of $\Delta x_i(t)$'s on the fraction of output allocated to a given investment $\hat{s}_i^x(t)$. For physical capital, we take its increase to be the result of a simple accumulation, i.e.:

$$\Delta K_i(t) = I_i^K(t) = \hat{s}_i^K(t)Y_i(t). \tag{5.15}$$

For human capital, we take a form similar to eq. (2.2) from the Manuelli-Seshadri model:

$$\Delta H_i(t) = z_i(t)\left(\hat{s}_i^H(t)Y_i(t)\right)^{\gamma_1} \left(\left(1 - \hat{n}_i(t)\right) H_i(t)\right)^{\gamma_2}, \tag{5.16}$$

where: $z_i(t)$ – the efficiency of human capital accumulation, $\hat{n}_i(t)$ – fraction of time devoted to production, γ_1, γ_2 – parameters. The increase in human capital depends on the human capital investment $I_i^H(t)$, but is not equivalent to a simple accumulation process as for physical capital.

Finally, for technology, we take the following equation, analogous to eq. (3.29), including an own-research term and a diffusion term:

$$\Delta A_i(t) = \left(T(t) - A_i(t)\right) r_i^A(t) + \mu_i(t)\left(\hat{s}_i^A(t)\right)^{\eta} \left(A_i(t)\right)^{\gamma_A}, \tag{5.17}$$

where: $T(t)$ – the technology frontier, $r_i^A(t)$ – the efficiency of implementation of the frontier technology[7], $\mu_i(t)$ reflects the size of the firm's own technological improvement, related to the company's R&D expenditures $I_i^A(t) = \hat{s}_i^A(t)Y_i(t)$, η, γ_A – parameters. The frontier technology $T(t)$ is assumed to be growing exponentially with a growth rate σ:

$$T(t) = T(0)e^{\sigma t}. \tag{5.18}$$

The vital part of the model is the choice of the product division variables $\hat{s}_i^x(t)$ and the time division variable $\hat{n}_i(t)$ for all firms $i = 1, \ldots, N$ and all moments of time t. Each firm should choose such values of the variables that

[7] In an ideal case, one can imagine a complete imitation of the leader's technology, resulting in an immediate catch-up with the frontier technology $A_i(t) \to T(t)$. More realistically, technology diffusion from the leader country to the firm under consideration will be much slower, which is reflected in the value of $r_i^A(t)$ at a given time t. Thus $0 < r_i^A(t) < 1$.

maximize the utility of consumption of the owner(s) over the time horizon $[t,\, t+T_{max}]$:

$$U_i(t) = \sum_{t'=0}^{T_{max}} u(C_i(t+t'))e^{-\rho_i(t+t')t'}, \tag{5.19}$$

where $U_i(t)$ – sum of discounted utilities of consumption over the horizon $[t,\, t+T_{max}]$, $\rho_i(t+t')$ – the discount rate of firm i at time $t+t'$ and:

$$u(C_i(t+t')) = \frac{(C_i(t+t'))^{1-\epsilon} - 1}{1-\epsilon} \tag{5.20}$$

is a CIES-type utility function, ϵ being the inverse of the intertemporal elasticity of substitution[8].

We now assume that the owners of the firms do not know the exact values of the variables $\rho_i(t)$, $\delta_i^K(t)$, $\delta_i^H(t)$, $\delta_i^A(t)$, $z_i(t)$, $r_i^A(t)$ and $\mu_i(t)$ – they are aware only of their expected values, denoted by $\tilde{\rho}$, $\tilde{\delta}^K$, $\tilde{\delta}^H$, $\tilde{\delta}^A$, $\tilde{z}$, $\tilde{r}^A$ and $\tilde{\mu}$, respectively[9]. This introduces *randomness* (*uncertainty*) to the model.

Therefore, the owners do not know the *exact dynamics* of the relevant quantities, only the *expected dynamics*, given by the set of equations (5.21)-(5.24). For each time t, the owner(s) of the i-th firm calculates the expected dynamics for the moments $t' = 1, 2, \ldots, T_{max}$ succeeding the time t. Thus, we denote by $\tilde{X}_i(t,t')$ (where $X = Y, K, H, A, C, I^K, I^H, I^A$) the expected level of the variable X for the time moment $t+t'$, prediction being made at time t. In addition, we have $\tilde{X}_i(t,0) \equiv X_i(t)$. Let us introduce the following definition. The expected dynamics of the i-th firm $E_i[t, T_{max}, \hat{s}_i^K(t), \hat{s}_i^H(t), \hat{s}_i^A(t), \hat{n}_i(t)]$ for the time horizon $[t+1, T_{max}]$ is the set of trajectories $\{\tilde{Y}_i(t,1)$, $\tilde{C}_i(t,1)$, $\tilde{K}_i(t,1)$, $\tilde{H}_i(t,1)$, $\tilde{A}_i(t,1), \ldots, \tilde{Y}_i(t,T_{max})$, $\tilde{C}_i(t,T_{max})$, $\tilde{K}_i(t,T_{max})$, $\tilde{H}_i(t,T_{max})$, $\tilde{A}_i(t,T_{max})\}$ of expected levels of the state variables of the model for the time horizon $[t+1, T_{max}]$, with decision variables set to $\hat{s}_i^K(t)$, $\hat{s}_i^H(t)$, $\hat{s}_i^A(t), \hat{n}_i(t)$, given by the following set of equations (5.21)-(5.24):

$$\begin{aligned} \tilde{Y}_i(t,t') &= \tilde{C}_i(t,t') + \tilde{I}_i^K(t,t') + \tilde{I}_i^H(t,t') + \tilde{I}_i^A(t,t') = \\ &= (1 - \hat{s}_i^K(t) - \hat{s}_i^H(t) - \hat{s}_i^A(t))\tilde{Y}_i(t,t') + \\ &+ \hat{s}_i^K(t)\tilde{Y}_i(t,t') + \hat{s}_i^H(t)\tilde{Y}_i(t,t') + \hat{s}_i^A(t)\tilde{Y}_i(t,t'), \end{aligned} \tag{5.21}$$

$$\tilde{K}_i(t,t'+1) = \left(1 - \tilde{\delta}^K\right)\tilde{K}_i(t,t') + \hat{s}_i^K(t)\tilde{Y}_i(t,t'), \tag{5.22}$$

$$\begin{aligned} \tilde{H}_i(t,t'+1) &= \left(1 - \tilde{\delta}^H\right)\tilde{H}_i(t,t') + \\ &+ \tilde{z}\left(\hat{s}_i^H(t)\tilde{Y}_i(t,t')\right)^{\gamma_1}\left((1-\hat{n}_i(t))\,\tilde{H}_i(t,t')\right)^{\gamma_2}, \end{aligned} \tag{5.23}$$

$$\begin{aligned} \tilde{A}_i(t,t'+1) &= \tilde{A}_i(t,t') + \left(T(t+t') - \tilde{A}_i(t,t')\right)\tilde{r}^A + \\ &+ \tilde{\mu}\left(\hat{s}_i^A(t)\right)^{\eta}\left(\tilde{A}_i(t,t')\right)^{\gamma_A}. \end{aligned} \tag{5.24}$$

The above equations allow us to find an interesting interpretation for the parameter T_{max}. As t' is increased towards T_{max}, the values $X(t,t')$ become increasingly uncertain. If the level of uncertainty for a given economy is high, the predictions become very inaccurate even for a moderate t'. Similarly, if the degree of uncertainty is low, the predictions can be accurate even for a relatively large t'. Therefore, the parameter T_{max} measures the level of uncertainty of an economy – the higher T_{max}, the lower the uncertainty.

For a given time t and uncertainty parameter T_{max}, the owner(s) of the i-th firm choose the optimal division of the product $Y_i(t)$ between consumption and investments, expressed by the decision variables $\hat{s}_i^K(t), \hat{s}_i^H(t), \hat{s}_i^A(t), \hat{n}_i(t)$.

In other words, from all possible sets of trajectories $E_i[t, T_{max}, \hat{s}_i^K(t), \hat{s}_i^H(t), \hat{s}_i^A(t), \hat{n}_i(t)]$ the i-th firm's owners choose the trajectory that should lead[10] to the highest possible utility of consumption. Now, we introduce another definition. The optimal division of the product $Y_i(t)$ is the set of values of the decision variables $\hat{s}_i^K(t), \hat{s}_i^H(t), \hat{s}_i^A(t), \hat{n}_i(t)$ that leads to the expected dynamics $E_i[t, T_{max}, \hat{s}_i^K(t), \hat{s}_i^H(t), \hat{s}_i^A(t), \hat{n}_i(t)]$ which assures the highest expectation value of the utility of consumption over the time horizon $[t, t+T_{max}]$:

$$\tilde{U}_i(t) = \sum_{t'=0}^{T_{max}} u(\tilde{C}_i(t,t'))e^{-\tilde{\rho}t'}. \tag{5.25}$$

Thus, the optimal division of the product $Y_i(t)$ leads to the notion of the optimal expected dynamics. The optimal expected dynamics of the i-th firm for the time horizon $[t, T_{max}]$ $E_i^{opt}[t, T_{max}, \hat{s}_i^K(t), \hat{s}_i^H(t), \hat{s}_i^A(t), \hat{n}_i(t)]$ is the expected dynamics $E_i[t, T_{max}, \hat{s}_i^K(t), \hat{s}_i^H(t), \hat{s}_i^A(t), \hat{n}_i(t)]$ with the decision variables set to their optimal values.

At each time step, the actual dynamics of the economy differs from the expected dynamics, because in reality the dynamics of the state variables is given by the set of equations (5.11)-(5.17) and not (5.21)-(5.24). Therefore, there are differences between the owners' expectations and the actual values of production, physical capital, human capital, technology and consumption. These differences cause that the owners' decision might not be optimal *ex post*.

As an example, let us consider a few time steps in the evolution of some economy. We choose some firm i. The initial conditions are the levels of capitals $K_i(0)$, $H_i(0)$ and technology $A_i(0)$ which lead to some value of production $Y_i(0)$. The owner(s) chooses the optimal division of the product $Y_i(0)$ represented by the values $\hat{s}_i^K(0), \hat{s}_i^H(0), \hat{s}_i^A(0)$ and $\hat{n}_i(0)$. This choice will be optimal if the actual dynamics will closely follow the optimal expected dynamics $E_i^{opt}[0, T_{max}, \hat{s}_i^K(0), \hat{s}_i^H(0), \hat{s}_i^A(0), \hat{n}_i(0)]$. However, at $t = 1$ the state variables differ from their expected values $Y_i(1) \neq \tilde{Y}_i(0,1)$, $K_i(1) \neq \tilde{K}_i(0,1)$ etc. This entails a correction of the optimal division of the product, leading to the optimal expected dynamics $E_i^{opt}[1, T_{max}, \hat{s}_i^K(1), \hat{s}_i^H(1), \hat{s}_i^A(1), \hat{n}_i(1)]$. In other words, if the actual dynamics is at time t different from the expected dynamics, we have $\tilde{X}_i(t,1) \neq X_i(t+1)$ and $\tilde{X}_i(t,t') \neq \tilde{X}_i(t+1, t'-1)$.

The model can easily be simulated with a direct Monte Carlo method. At every time step, the optimal division of the product and the optimal expected dynamics for every firm can be calculated. Then, one simulates the actual dynamics and one repeats the computation scheme for the next time step. Thus, for every time t one can find the aggregate (or, equivalently, average) values of the state variables and their growth rates. As we will show in the next subsection, the model can lead to a non-trivial dynamics and interesting economic conclusions.

[10] In the case that all random variables take their expected values.

5.2.2 Properties of the model

To examine the properties of the model, we assume the parameter values given in Tabs. 5.3 (parameters that are not random variables) and 5.4 (parameters of random variable distributions).

Table 5.3: Parameters of the model used in simulations.

α	γ_1	γ_2	σ	η	γ_A	T_{max}	ϵ
0.315	0.63	0.3	0.0208	0.5	1.5	20	1

Table 5.4: Parameters of random variable distributions (with standard deviation equal to $\nu = 20\%$ of the mean).

	δ_i^K	δ_i^H	z_i	ρ_i	r_i^A	μ_i
mean	0.06	0.018	0.361	0.07	0.02	0.1
std.dev.	0.012	0.0036	0.0722	0.014	0.004	0.02

The parameters α, γ_1, γ_2 and the expected values of the random variables $\delta_i^K(t)$, $\delta_i^H(t)$, $z_i(t)$ and ρ_i ($i = 1, 2, \ldots, N$, with each firm expecting the same value of these variables) have been chosen at the same levels as the analogous variables in the Manuelli-Seshadri model. The value of σ has been taken at its empirical value for the United States for years 1981-1999. The remaining parameter values have been chosen arbitrarily, assuming only that the two terms in eq. (5.17) are of similar importance (technology diffusion and the firms' own R&D contribute equally to technological progress) and that there are increasing returns to scale in research activity. We also assume that the time horizon T_{max} equals 20 years[11]. However, the influence of this parameter will also be investigated (by explicitly comparing the predictions of the model for $T_{max} = 15, 20$ and 25 years).

It is worth to mention that other sets of parameters have also been examined. We have concluded that there is no qualitative change of the model dynamics, provided the values are chosen in an economically sensible manner. Thus, the assumed parameter set can be thought of as a typical set of parameters for this model.

Fig. 5.5 shows the influence of the number of firms (sectors) on the dynamics of production. A characteristic feature of the model (similar to the Solow-Swan simulational model) is that for large N the trajectories of all the factors of production, of consumption and production are *smooth*, with *statistical fluctuations averaged out*. The picture of the economy is one of a set of *representative agents*, with, by definition, identical characteristics. Therefore, one could call such a model an extension of the Ramsey-Cass-Koopmans model to include human capital accumulation and technological progress. Further justification of such a view on the model in the limit of large N will be given in a moment.

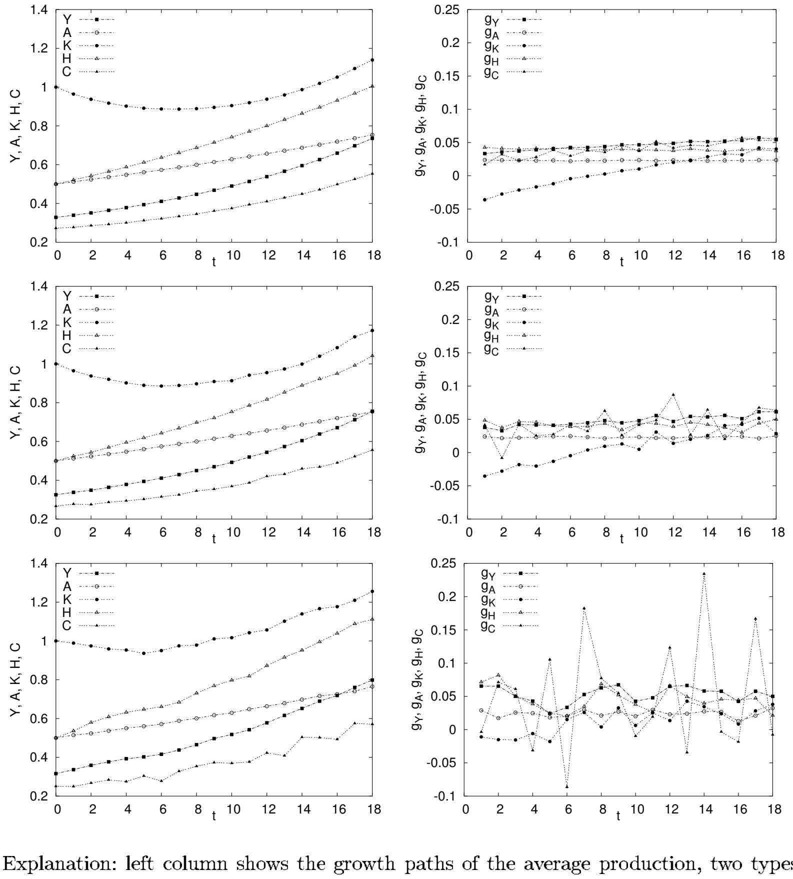
On the other hand, *for smaller N the statistical fluctuations play a bigger role*. The size of the fluctuations is determined by the standard deviations of

[11]Having chosen the above values for the parameters of the model, we have also implicitly set the time unit to 1 time step = 1 year.

the relevant variables. The plots of Fig. 5.5 for $N = 10$ and $N = 1$ are much more similar to empirical data plots, making this regime of the model very interesting to examine. While it explicitly corresponds to a small amount of firms, such an interpretation is not mandatory. In neoclassical models of economic growth, one often considers economies with just one product. The assumption of one aggregate product is not to be taken literally – it rather reflects the fact that from the economic point of view what really counts is the total value of produced goods and not the fact that there are many firms who produce many products.

However, the presence of statistical fluctuations in the time series of virtually all quantities of interest is an empirical fact and it may be interesting to explore the properties of these fluctuations, since it can lead to a better comprehension of the underlying mechanisms. Thus, the regime of small N can be of interest and we can then interpret N rather as the number of sectors in the economy under analysis. Another reason for this is the intuition that the dynamics of firms from the same sector can be highly correlated,

Figure 5.5: **The dependence of the production dynamics on the number of firms (sectors)** N



Explanation: left column shows the growth paths of the average production, two types of capital, technology and consumption for 100, 10 and 1 firm. Right column shows the growth rates of the respective quantities (g_x, where x is the quantity under consideration). Parameter $\nu = 0.2$.

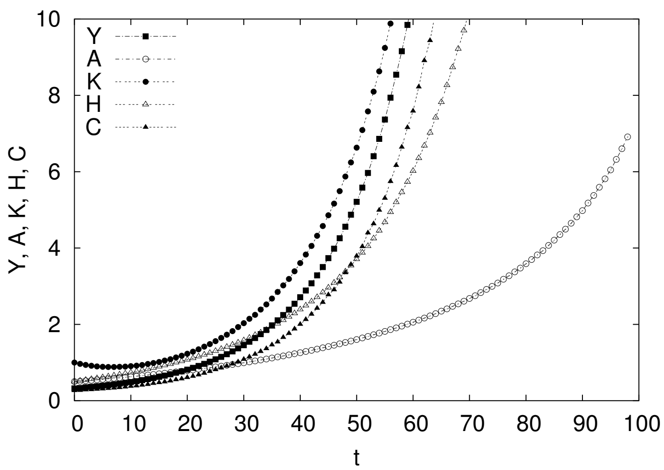
e.g. a cold summer will lead to lower sales of all companies who produce beer – in this way the *fluctuation* (cold summer) leads to the same effect for all beer companies and all of them can be described by just one agent in our model.

Another interesting feature of the model that can be seen in Fig. 5.5 is that for small N the fluctuations are often *cyclical*. This results from the fact that any statistical fluctuation leads to some feedback on the part of the company, e.g. higher effective depreciation of physical capital can lead to an increased investment in physical capital – on the plot for the growth rate of physical capital one can see then a cyclical behaviour.

Figure 5.6 shows the dynamics of the economy (for $N = 100$ firms) in the long-run. One can observe that the growth rates of production, physical capital and human capital tend to equalize – the economy approaches a steady state (a balanced growth path), typical for neoclassical models of economic growth[12]. In this class of models, however, one usually solves for the balanced growth path, for which the growth rates of the relevant variables are equal. A solution for transitional dynamics – how the economy starts from arbitrary initial conditions and how it approaches the state of balanced growth – is often hard to find analytically. In simulational models of growth, on the other hand, transitional dynamics is no more difficult to find than the steady state, which is clearly another advantage of this paradigm of modelling.

It is worth adding that in the model under analysis, not all variables grow with the same growth rate (the growth rates of technology and human capital differ from each other and from the common growth rate of production, consumption and physical capital). This is a built-in feature of the model, resulting from the fact that there is no straightforward link between the dynamics of technology, human capital and production (whereas the dynamics of physical capital and consumption is related explicitly to the dynamics of production).

Figure 5.6: **The growth paths of production, consumption, physical capital, human capital, technology (up) and their growth rates (down) in the long run.** $N = 100$ **firms,** $\nu = 0.2$



[12]For small N the statistical fluctuations are too large to allow a balanced growth type of behaviour.

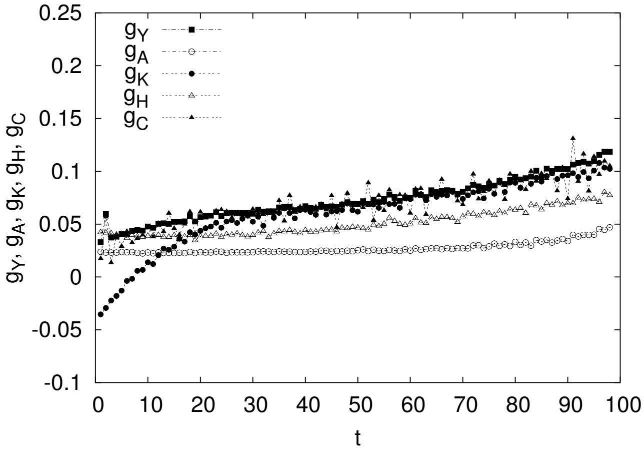
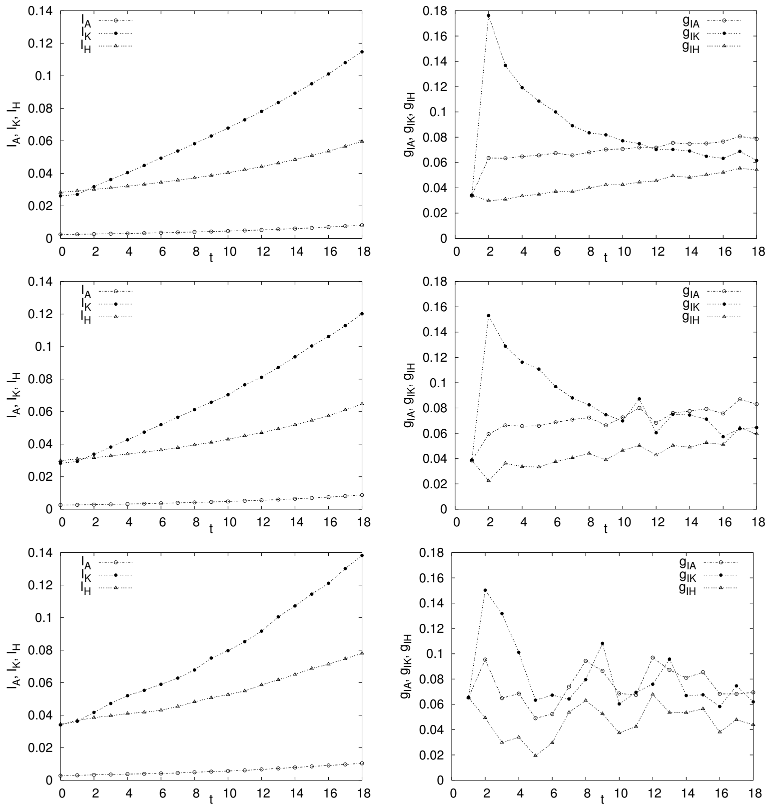


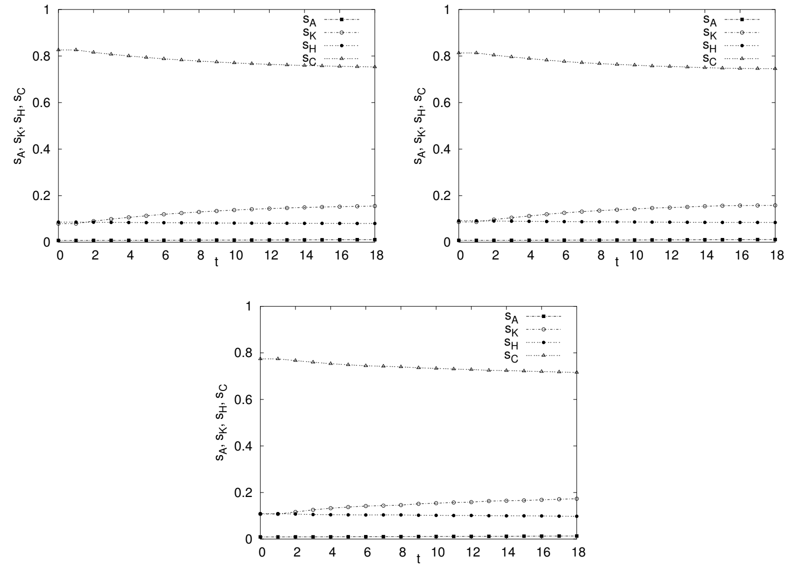
Fig. 5.7 shows the influence of the number of firms (sectors) on the investment dynamics. Similarly to the dynamics of production and the factors of production, the fluctuations of investments and their growth rates are the largest in the regime of small N. At the start of the period of analysis one can again see the transitional dynamics associated with the choice of the initial conditions.

Figure 5.7: **The dependence of the investment dynamics on the number of firms (sectors)**



Explanation: left column shows the growth paths of investment in physical capital, human capital and technology for 100, 10 and 1 firm. Right column shows the growth rates of investments g_{Ix} (where Ix denotes investment in x). Parameter $\nu = 0.2$.

Figure 5.8: **The dependence of the division of output on the number of firms for 100, 10 and 1 firm, respectively**

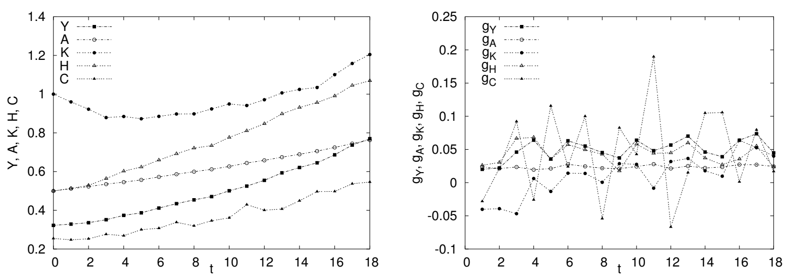


Explanation: The plots show the average fraction of output devoted to consumption and investments in physical capital, human capital and technology. Parameter $\nu = 0.2$.

Fig. 5.8 shows that the number of firms has no visible influence on the division of output between consumption and investments. Every firm divides its output depending on its discount rate and its resources at a given time. However, the levels of these variables are comparable and therefore the division of output is similar for all firms. A typical effect for all firms is that the investment rate increases over time. This is due to the increase in the stock of knowledge (human capital and technology) – to maximize the discounted sum of future utility it is profitable to invest increasingly more in physical capital.

In Figure 5.9 we show the dynamics of the basic variables of the model in the case of $N = 10$ firms. The trajectories are much less smooth and

Figure 5.9: **The dynamics of the model variables and their growth rates for $\nu = 0.5$ and $N = 10$ firms**



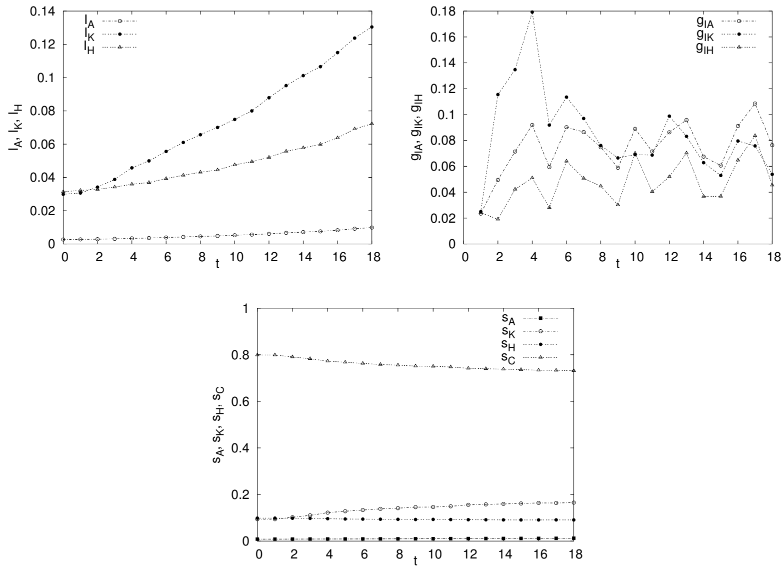
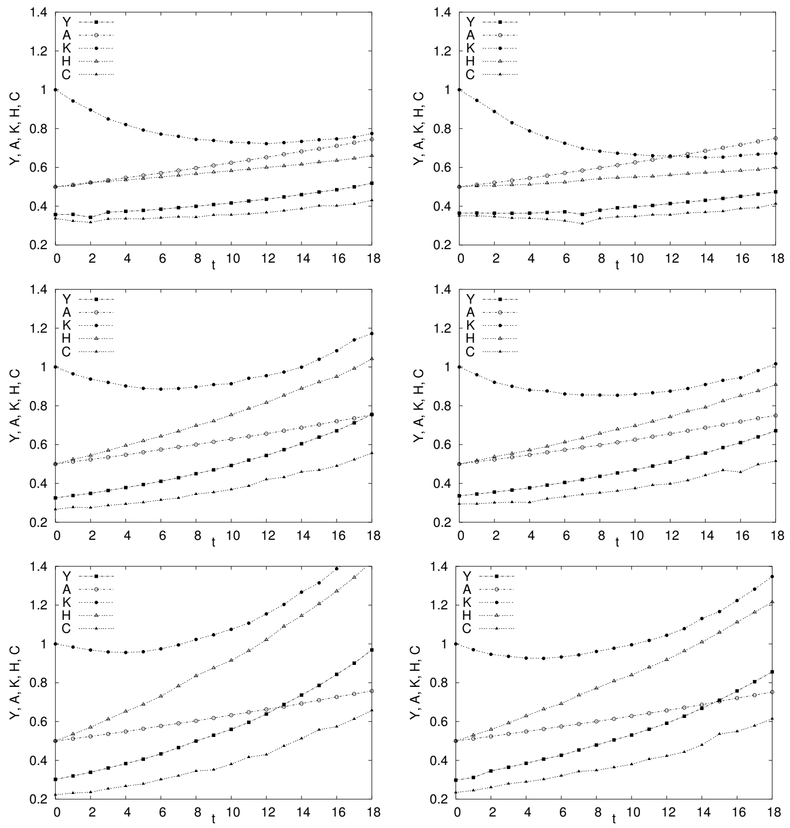


Figure 5.10: **The influence of the horizon T_{max} and the discount rate ρ on the dynamics of production, consumption, physical capital, human capital and technology**



Explanation: left column $\tilde{\rho} = 0,07$, right column $\tilde{\rho} = 0,10$. Top row $T_{max} = 15$, middle row $T_{max} = 20$, bottom row $T_{max} = 25$. $N = 10$ firms. Parameter $\nu = 0.2$.

the growth rates fluctuate with approximately five times larger amplitude[13]. The division of output is only slightly affected. The fluctuations are not large enough to threaten the stability of the economy. The stability would only be violated if one of the factors of production fell down to zero – otherwise an adjustment can always be made by a firm to compensate for even a large fluctuation.

Finally, we consider the influence of the discount rate and the length of the horizon T_{max} (Fig. 5.10). We analyze three values of the parameter T_{max} – 15, 20 and 25. For $T_{max} = 15$ a larger fraction of production is consumed and the investment rate is lower than for $T_{max} = 20$ or 25. As a consequence, the growth rates of the economy are lower. A similar effect on the economy is exerted by high values of the discount rate ($\rho = 0.1$ on the right side of the plot). In such case, the later periods are less important for the owner(s) of the firm and they prefer current consumption to investments, leading to an increase in future consumption. For large values of T_{max} or for small values of ρ, future consumption is relatively more important than current consumption and the growth rate of the economy is higher.

It may seem to be unreasonable why the length of the planning horizon should influence the growth rates. However, as we have already stated, under uncertainty T_{max} can be interpreted as the longest period over which the owners of the firm can reliably foresee the dynamics of the economy. If the uncertainty is large, the owner(s) can plan only for a few periods, for $t' > T_{max}$ it is not possible to predict anything. For smaller uncertainty, the owner(s) can look into a more distant future and thus it is profitable for them to invest more and contribute to faster growth.

The link between uncertainty and growth can be understood also in the following way. If the uncertainty is large, a lot of owners' decisions is non-optimal and the growth is slower. In the case of smaller uncertainty more decisions turn out to be optimal or near-optimal and the growth is faster.

In this way, the parameter T_{max} captures a similar property of the economy as the parameter ν – the ratio of the standard deviation to the mean for the quantities given in Table 5.4. However, it is not the same as the latter – it rather reflects the owners' sensitivity to uncertainty. Still, it is certainly one of the most important parameters from the point of view of empirical research based upon the model.

5.2.3 Empirical analysis for the OECD countries

The empirical analysis for the OECD countries in 1981-1999 was performed in the following way. The parameter values were taken from Tab. 5.3. The expected value of the parameter z was chosen at 0.361 (in accordance with the Manuelli-Seshadri model) and of the depreciation rate of human capital at 0.04, which leads to ca. 1-percent growth of human capital stock that we assumed in the previous chapter.

The values of the remaining parameters of random variables from Tab. 5.4 will be calibrated. The calibration procedure will consist in finding such parameter values that the model and empirical physical capital p.c., technology level and production p.c. (GDP p.c.) agree in the best possible way. The measure of this agreement is again the mean relative error MRE, defined by (3.32). The standard deviation was chosen at $v = 20\%$ of the respective mean and the planning horizon is $T = 20$ years. The simulations were performed for $N = 10$ firms (sectors).

[13]Compare with the middle rows of Fig. 5.5, 5.7 and the upper right plot of Fig. 5.8).

For each country we calibrate the discount rate (which determines the growth rate of GDP p.c.), depreciation rate of physical capital p.c. and the parameters μ and r^A (technology parameters).

In the case of USA, the technology leader, the parameter μ was found to be 0.18 and this value was chosen for the other countries, for which only the diffusion parameter r^A was calibrated (in USA $r^A \equiv 0$, by definition).

The results of the calibration procedure, together with mean relative errors of the calibration of physical capital, production and technology, are shown in Tab. 5.5. We observe the following regularities:

Table 5.5: **The results of the calibration procedure and the mean relative error for 28 OECD countries**

Country	$\tilde{\delta}^K$	$\tilde{\rho}$	$\tilde{r}^A$	MRE^K	MRE^Y	MRE^A
AUS	0.16	0.08	0.02	0.068	0.033	0.017
AUT	0.14	0.08	0.02	0.049	0.035	0.037
BEL	0.21	0.10	0.03	0.075	0.023	0.024
CAN	0.19	0.10	0.02	0.092	0.058	0.045
CZE	0.22	0.08	0.03	0.276	0.038	0.064
DEN	0.23	0.10	0.02	0.075	0.028	0.030
FIN	0.21	0.08	0.04	0.237	0.067	0.026
FRA	0.21	0.09	0.02	0.094	0.056	0.029
GER	0.16	0.08	0.02	0.069	0.030	0.024
GRE	0.17	0.06	0.02	0.050	0.013	0.070
HUN	0.27	0.04	0.01	0.120	0.082	0.129
IRL	0.28	0.06	0.06	0.118	0.062	0.067
ISL	0.20	0.10	0.06	0.081	0.031	0.029
ITA	0.21	0.09	0.05	0.079	0.054	0.026
JPN	0.07	0.09	0.04	0.087	0.049	0.026
KOR	0.00	0.01	0.02	0.130	0.044	0.042
MEX	0.21	0.04	0.01	0.193	0.077	0.107
NED	0.18	0.09	0.02	0.038	0.019	0.021
NZL	0.18	0.07	0.01	0.087	0.032	0.035
NOR	0.13	0.10	0.06	0.082	0.035	0.039
POL	0.26	0.04	0.01	0.251	0.074	0.117
POR	0.13	0.02	0.02	0.068	0.041	0.041
SPA	0.20	0.05	0.02	0.092	0.039	0.033
SWE	0.23	0.10	0.01	0.158	0.060	0.020
SWI	0.12	0.12	0.00	0.092	0.035	0.025
TUR	0.13	0.00	0.004	0.085	0.112	0.034
UK	0.22	0.09	0.02	0.094	0.050	0.024
USA	0.20	0.12	–	0.060	0.043	0.036

- The values of the physical capital depreciation rate are much higher than the level that is usually given in the literature – 6-7%. It results from the fact that we do not take taxation into account and thus the growth rate of physical capital that results from optimal division of production between consumption and investments is much too high if the depreciation rate is only 6-7%. Therefore, the calibration procedure requires to raise the expected value of the physical capital depreciation rate and interpret it as the measure of depreciation *and* taxation.

5

- The discount rate is high in the richest countries (USA, Canada, Denmark, Iceland, Norway, Sweden, Switzerland), and rather low in the relatively poorest countries (Greece, Hungary, South Korea, Mexico, Poland, Portugal, Turkey). Hence, the model confirms that the higher the wealth of a country, the smaller the tendency to postpone consumption (in favour of investments that raise future consumption), and the poorer the country, the higher this tendency to raise future consumption.

- The diffusion parameter $\tilde{r}^A$ takes the highest values for countries, where the diffusion parameter was high in the models that we investigated earlier (Ireland, Iceland, Italy, Norway). The smallest values are observed in the countries where we got the lowest values or negative ones (Hungary, Mexico, Poland, Sweden, Switzerland, Turkey). The agreement between the conclusions from these models means that there really are countries, where technology diffusion effects play an important role and a group of countries where they are relatively less important.

- The mean relative errors are the highest for physical capital, since the empirical time series for physical capital are the most irregular. The errors of GDP p.c. and technology calibration are much smaller. Generally, we have also smaller errors for the wealthier countries, where the empirical fluctuations are smaller.

According to the model, the consumption rate in 1981 fluctuated between 67.5% and 78% (Tab. 5.6). The remaining part of production was invested. The majority of investments went to physical capital – from 14.1% to 24.2% of production. The investments in human capital took 3.1% to 7.2% of production and in technology 1.2% to 4.5% of GDP. In 1999 the consumption rate was lower than in 1981 in all of the countries and equalled 62.8% to 74.1%. As a consequence, the investment rate increased, especially the investments in technology went up (to between 2.1% and 7.4%) and also in human capital (up to 5.7% to 10.5%). The physical capital investment rate grew in most of the countries, but it shrank in some of them, ranging in 1999 from 13% to 24.7% of production.

We should emphasize that the obtained values of investments are the optimal values. In the real-world, we always observe some deviations from these optimal rates and thus the sum of discounted utilities of consumption can be smaller than the maximal possible value.

It is interesting to examine the fluctuations in the growth rates of the analyzed quantities. The minimal and maximal values are gathered in Tab. 5.7. The amplitudes of fluctuations are rather similar to the amplitudes in the empirical time series. For example, the growth rates of production range for most of the countries between ca. 1% and ca. 5%. The largest fluctuations

Table 5.6: **The division of product between investments and consumption for 1981 and 1999**

Country	s^A 1981	s^K 1981	s^H 1981	s^C 1981	s^A 1999	s^K 1999	s^H 1999	s^C 1999
AUS	2.8	18.6	5.8	72.8	4.7	19.8	7.2	68.3
AUT	2.9	18.6	5.9	72.6	4.9	18.9	7.4	68.8
BEL	2.8	21.1	5.1	71.0	4.7	20.8	6.5	68.0

CAN	3.0	18.5	5.2	73.2	5.0	20.0	6.5	68.5
CZE	2.3	17.9	4.3	75.5	3.6	22.2	6.6	67.7
DEN	3.4	22.5	5.4	68.7	5.6	21.7	6.9	65.8
FIN	2.2	20.6	5.8	71.5	4.3	21.9	7.2	66.7
FRA	3.1	20.2	5.4	71.2	5.3	21.1	6.9	66.7
GER	3.0	19.9	5.8	71.3	5.0	19.8	7.3	67.9
GRE	1.7	20.1	5.0	73.1	3.6	21.2	7.2	67.9
HUN	1.6	23.6	3.8	71.0	2.9	24.7	6.1	66.3
IRL	1.5	24.1	6.8	67.5	4.0	24.7	8.4	62.8
ISL	2.0	19.4	5.3	73.2	3.6	20.7	6.5	69.2
ITA	2.0	20.5	5.4	72.1	3.9	21.6	6.8	67.8
JPN	2.0	14.1	5.9	78.0	3.8	14.7	7.4	74.1
KOR	1.2	18.7	7.2	73.0	3.4	13.0	10.5	73.0
MEX	1.5	22.1	4.0	72.4	2.8	23.2	6.3	67.7
NED	3.0	20.4	5.2	71.3	4.8	20.2	6.8	68.2
NZL	3.1	20.0	5.6	71.3	4.8	20.8	7.1	67.3
NOR	1.8	16.9	5.5	75.7	3.5	17.7	6.8	72.0
POL	1.2	24.2	3.1	71.5	2.4	24.6	5.7	67.3
POR	1.7	20.1	7.1	71.2	3.9	21.1	9.2	65.8
SPA	2.3	22.5	6.0	69.2	4.4	22.6	7.9	65.2
SWE	3.3	21.0	4.6	71.2	5.1	21.4	6.1	67.3
SWI	3.8	14.7	4.7	76.8	5.7	16.2	6.1	72.0
TUR	1.4	23.1	4.1	71.3	2.1	20.7	6.4	70.7
UK	2.9	22.3	5.0	69.8	4.8	21.8	6.6	66.8
USA	4.5	18.9	5.1	71.5	7.4	19.5	6.6	66.5

Table 5.7: **Minimal and maximal growth rates of production, physical and human capital, technology and consumption**

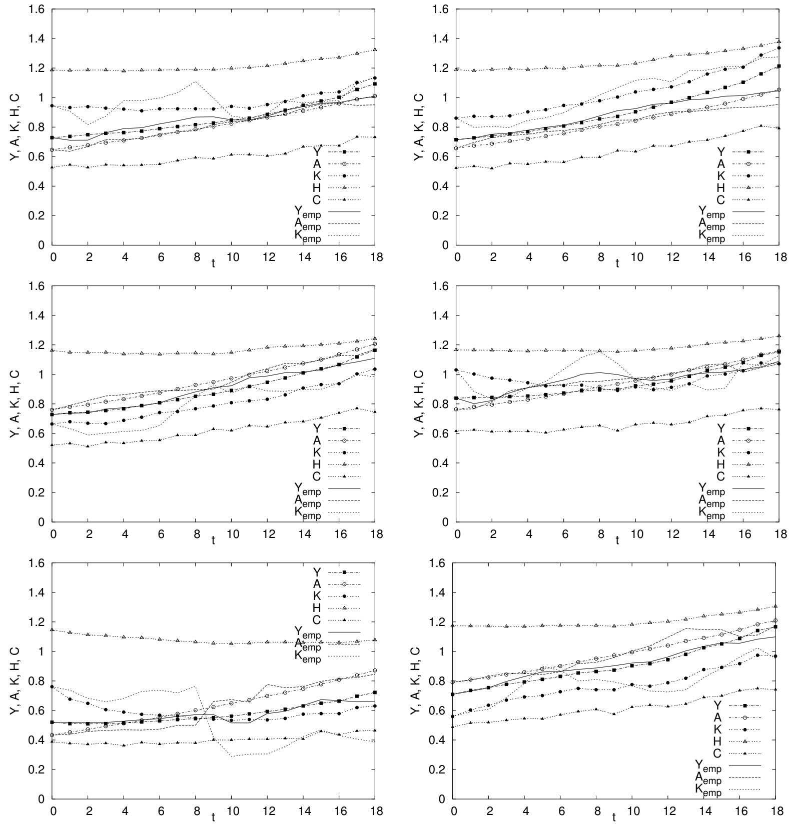
Country	min g_Y	max g_Y	min g_A	max g_A	min g_K	max g_K	min g_H	max g_H	min g_C	max g_C
AUS	0.6	5.3	2.2	3.0	-1.3	6.1	-0.6	2.1	-3.3	8.9
AUT	1.6	5.2	2.0	3.3	-0.2	6.8	-0.6	2.0	-2.7	6.4
BEL	0.4	4.9	1.9	3.3	-1.5	7.1	-1.1	1.6	-3.9	6.9
CAN	0.4	4.4	1.8	3.1	-2.8	5.7	-0.5	1.6	-5.3	6.8
CZE	-1.8	5.0	3.2	4.5	-11.1	7.2	-1.6	1.0	-4.9	12.9
DEN	1.1	5.0	1.8	3.2	-1.4	7.8	-0.4	1.8	-5.4	8.4
FIN	1.0	6.2	2.5	3.8	-4.4	7.3	-0.6	2.0	-4.0	9.9
FRA	0.3	5.0	2.0	3.5	-3.0	6.8	-0.4	2.1	-6.2	5.1
GER	1.6	5.2	2.0	3.3	0.3	7.3	-0.7	1.9	-2.8	7.8
GRE	1.1	5.0	3.3	4.3	-2.5	6.1	-1.2	1.5	-5.0	4.9
HUN	-1.6	3.3	2.8	3.2	-7.2	5.4	-2.4	0.0	-4.6	5.4
IRL	3.0	6.9	3.9	10.2	-9.6	9.7	-0.9	2.1	-1.4	11.4
ISL	0.9	4.8	2.6	3.4	-2.8	5.8	-0.8	2.3	-3.3	5.2
ITA	1.2	5.2	2.8	3.9	-3.1	5.9	-1.0	1.9	-1.3	6.0
JPN	2.9	4.5	2.4	4.1	1.5	4.1	-0.5	2.0	0.4	5.9
KOR	5.8	9.7	4.5	9.0	6.6	15.1	-1.5	3.4	3.4	13.2
MEX	-0.8	3.2	2.8	3.3	-4.8	4.2	-2.2	-0.1	-3.7	5.4
NED	1.3	4.2	1.7	3.0	1.0	5.9	-1.1	1.7	-2.6	6.6
NZL	0.3	4.1	1.7	2.7	-2.2	5.9	-0.9	1.7	-4.0	5.6
NOR	2.0	4.9	2.7	4.0	-0.5	5.9	-0.8	2.0	-1.6	6.6
POL	-0.4	3.2	3.1	3.8	-3.8	5.3	-2.9	-0.3	-4.3	5.9

POR	2.5	5.0	3.7	5.3	-1.6	5.3	-0.6	1.5	-3.5	12.1
SPA	1.7	4.9	3.0	3.5	-0.5	6.4	-1.1	1.7	-2.8	8.3
SWE	0.0	3.8	1.2	2.8	-2.2	5.8	-1.4	1.3	-3.7	5.6
SWI	0.3	3.9	1.4	2.7	-0.7	3.8	-0.7	2.0	-2.2	4.1
TUR	0.4	371	2.0	2.5	0.3	14.0	-2.5	0.4	-2.4	6.1
UK	0.8	4.4	1.7	3.1	-0.1	8.0	-1.2	1.7	-2.1	5.7
USA	0.4	5.4	1.5	3.5	-1.1	6.3	-0.3	2.3	-4.9	7.0

are observed for physical capital and consumption – there are periods with negative growth rates of these variables and also periods with growths of the order of a dozen percent or so. The smallest amplitude of fluctuations was obtained for human capital, however, we still observe both negative and positive values of the growth rates.

Figs. 5.11-5.15 show the model and empirical dynamics of the analyzed quantities for 28 OECD countries.

Figure 5.11: **The empirical and model dynamics of the analyzed quantities for Australia, Austria, Belgium, Canada, the Czech Republic and Denmark**



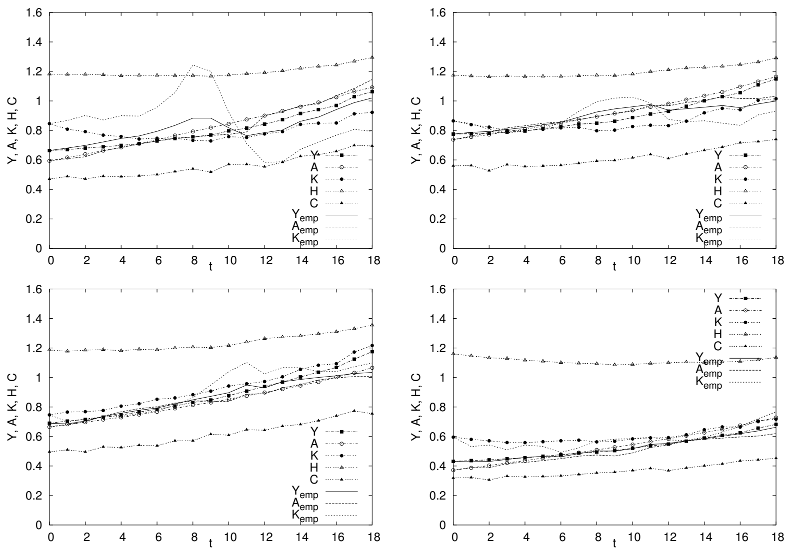
The agreement between the model and empirical data is rather good in the case of technology and GDP p.c. and much worse for physical capital. It results from the fact that the empirical dynamics of GDP p.c. and technology are quite regular, whereas the trajectories of physical capital are very irregular, which suggests that there are large statistical uncertainties related to the measurement of this quantity.

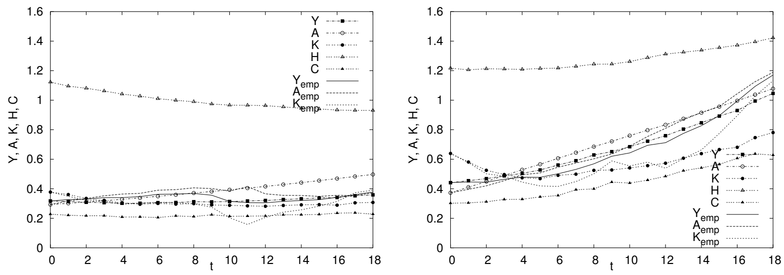
The considered model leads to arbitrary large fluctuations of the analyzed quantities – their amplitude depends on the chosen value of the parameter ν and the number of firms (sectors). The random character of these fluctuations means, however, that the measure of agreement – the mean relative error – can point that the error is rather large, hence sometimes a much better indicator would be a measure of the amplitude of fluctuations. This aspect of the model – the analysis of variability – was not considered in this book, but it can be an interesting direction for future research. What is more, the fluctuations observed in empirical time series can not be analyzed within the framework of *representative agents*, since they result from *differences* between the agents.

The method used in the empirical analysis (the calibration of GDP, physical capital and technology) makes it possible to draw conclusions about the dynamics of human capital in the analyzed period. We can distinguish countries where the human capital stock grew substantially during this period (Austria, Germany, Ireland, South Korea, Switzerland), grew moderately (Australia, Canada, Denmark, Finland, France, Iceland, Italy, Japan, The Netherlands, New Zealand, Norway, Portugal, Spain, USA), did not change (Belgium, Sweden, United Kingdom), shrank slightly (the Czech Republic, Greece) or decreased substantially (Hungary, Mexico, Poland, Turkey). It is, however, worth to emphasize that the countries with a decrease in human capital stock are the countries of its worst quality and lowest quantity, according to the Manuelli-Seshadri model. The conclusion from the analyzed model is that it was not profitable to invest in human capital in these countries – it was just more profitable to invest in physical capital and/or technology.

However, we do not take into account differences in the quality of schooling, the consideration of which could lead to new interesting conclusions and is thus a possible direction for further research.

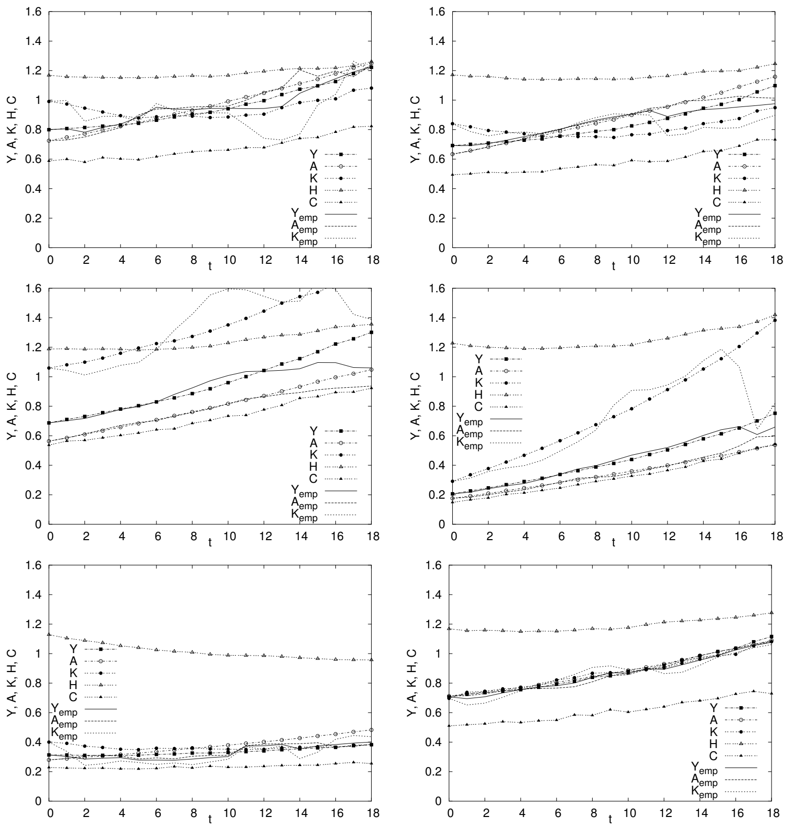
Figure 5.12: **The empirical and model dynamics of the analyzed quantities for Finland, France, Germany, Greece, Hungary and Ireland**





Also, the diffusion mechanism is rather too simplified in the model. The elasticity of the simulational approach consists in making it possible to investigate the conclusions from many different rules of behaviour of the agents – hence one can examine the consequences of a more complex diffusion mechanism, taking into account e.g. the effects of international trade in such a

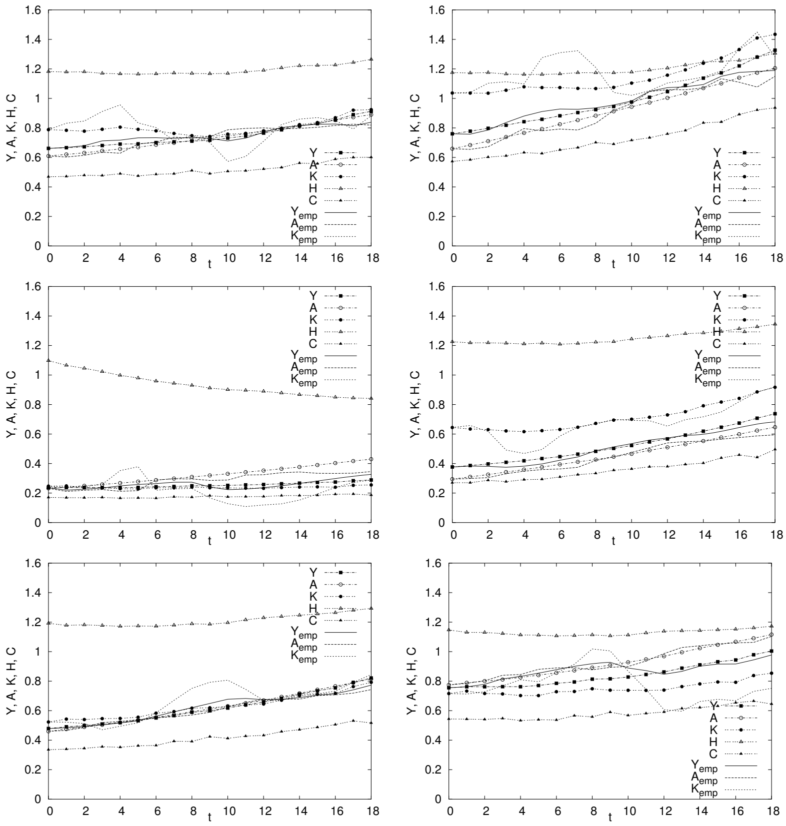
Figure 5.13: **The empirical and model dynamics of the analyzed quantities for Iceland, Italy, Japan, South Korea, Mexico and The Netherlands**



way that we would have diffusion effects between arbitrary countries and not only between the leader and other countries.

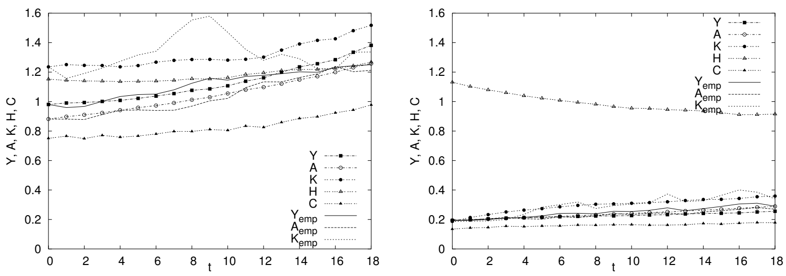
Another conclusion from the empirical analysis is that there are large differences in the values of the diffusion coefficient. In relatively poor countries this coefficient is quite low, which means that technology diffusion is

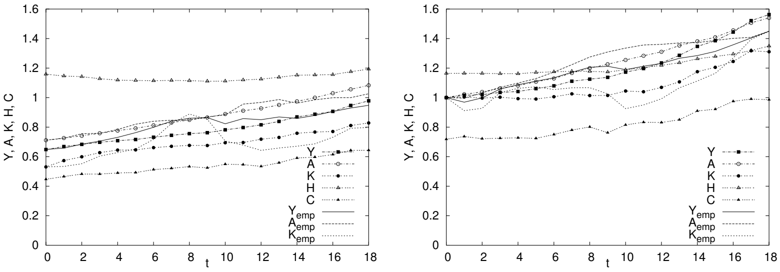
Figure 5.14: **The empirical and model dynamics of the analyzed quantities for New Zealand, Norway, Poland, Portugal, Spain and Sweden**



not as fast as it could be at higher efficiency. The key question for such countries is then: how to improve the efficiency of technology diffusion and thus catch up with the wealthier countries? The model also points that the R&D expenditures in the poorer countries as a fraction of GDP are smaller than their optimal value. Both of these effects (lower efficiency of technology diffusion and lower than optimal R&D expenditures) mean that the growth

Figure 5.15: **The empirical and model dynamics of the analyzed quantities for Switzerland, Turkey, United Kingdom and USA**





rate of technology is much smaller than the rate that is possible to achieve. Thus, the growth rate of GDP p.c. is also not as high as it could be.

Lastly, it is worth to emphasize that it is a very simplified picture of the real world. However, its main advantage is that it can be widely extended, which is not possible in the case of many neoclassical models. We also pointed to a few directions for future research. However, we could draw some interesting conclusions also from the present version of the model and we could confirm some of the intuitions about the essence of economic growth.

5.3 Summary

In this chapter the *heterogeneous agents* approach to the modelling of economic processes was considered. We analyzed two models, in which the effects of differences between the agents are important.

In the first model, every agent makes decisions about dividing the production between consumption and investments in three types of capital – physical, human and technological capital. We showed that in the limit of small diversity of agents the model reduces to a Solow-Swan-type model with three types of accumulated capital. In particular, we observe a balanced growth of production, consumption, physical capital, human capital and technology.

The second model is a Ramsey-Cass-Koopmans-type model with endogenous mechanisms of human capital accumulation and technology and the division of product between consumption and investments. In the economy described by this model, we also observe a balanced growth of production, physical capital and human capital. The steady-state consumption grows with a little lower rate of growth. Also, technology grows at a different rate because of the diffusion processes. We performed empirical analysis basing on the model and pointed to the directions for future research.

Conclusion

The aim of this book was to consider different ways of theoretical modelling of technological progress and human capital in the context of economic growth and to examine the implications of different interpretations of these notions. The most important hypothesis of the book was that human capital and technological progress are the most important determinants of economic growth.

The structure of the book was the following. In the first chapter we reviewed the world literature on the relationship between human capital, technological progress and economic growth. The second chapter was devoted to considerations about human capital, based on the Manuelli-Seshadri model. In the third chapter we analyzed the extension of this model to include the technological sector and we introduced two new models of technological progress with technology diffusion. The fourth chapter was devoted to the simulational approach to technological progress and technology diffusion. In the fifth chapter we introduced two simulational models of simple economies with physical capital, human capital and technology.

The most important results that were presented in the book are (in the order of appearance in the book):

- an application of the Manuelli-Seshadri model to estimate the quantity and quality of human capital in the OECD countries, with special attention to Poland and USA,
- a modification of the Manuelli-Seshadri model to include exogenous technological progress and technology diffusion, as well as empirical analysis of the modified model,
- an introduction of a technology dynamics equation with a diffusion term and its empirical verification,
- an introduction and empirical analysis of an endogenous model of technological progress with two research sectors – the technology diffusion sector and the implementation sector,
- an analysis and empirical verification of a simple simulational model of technological diffusion,
- a modification of the microeconomic evolution model, which consisted in an introduction of a mechanism of technology diffusion and empirical analysis of such model,
- an introduction of a simulational model of a simple economy with three types of capital,
- an introduction and empirical analysis of a simulational model of a simple economy with endogenous mechanisms of physical and human capital accumulation and technological progress,
- a proposal of a way to perform empirical analysis of simulational models,

- a demonstration of the advantages of the *heterogeneous agents* approach with respect to the *representative agents* approach – the standard approach in neoclassical growth theory,
- an exhibition of the fact that simulational models can reproduce the results of the neoclassical models (such as the Ramsey-Cass-Koopmans model) and in addition they can describe some of the real-world aspects of macroeconomic dynamics, which are absent in the neoclassical models.

The most important conclusions from the empirical analyses for the OECD countries are the following.

- Sweden is the country with the highest human capital stock p.c. Human capital resources in this country are higher than human capital resources of countries with higher average length of schooling. Thus, we can conclude that the quality of schooling and hence the quality of human capital is the highest in Sweden.
- Other countries with large human capital stocks p.c. are Finland, Norway and Canada. The highest quality of human capital was observed for Finland, The Netherlands and Denmark.
- The countries with lowest human capital resources p.c. are Portugal, Poland, Italy and Hungary. For Italy, it results from a short average length of schooling and for the other countries also from a low quality of schooling. Poland is a country of relatively long average length of schooling and very low quality of schooling – in countries with similar average lengths of schooling human capital stock is nearly twice as large as in Poland.
- One can distinguish a group of countries where human capital stock grew substantially in the analyzed period (Austria, Germany, Ireland, South Korea, Switzerland), a group of intermediate countries and a group where human capital stock decreased (the Czech Republic, Greece, Hungary, Mexico, Poland, Turkey). It is noticeable that these group consists of the poorest countries. To enable faster growth, it would be advisable for them to increase investments in human capital and the quality of schooling.
- The differences in human capital resources can not fully account for the observed cross-country differences in GDP p.c. Hence, we also have to take into account the differences in the levels of technology.
- The diffusion coefficients differ substantially among countries. The flow of technology is the most effective for Ireland, Iceland, Italy and Norway. These countries profited the most from technology diffusion in the analyzed period.
- We observe low efficiency of technology transfer to France, Germany, Sweden and Switzerland. These are the richest countries, which should be considered rather as the suppliers of technology than as receivers.
- Among the relatively poor countries, like Hungary, Mexico, Poland and Turkey, the diffusion of technology was the most important mechanism of technology growth, but the efficiency of this process was very low. Hence, increasing this efficiency could substantially accelerate their rate of economic growth.

- The diffusion coefficients do not depend on human capital resources – e.g. in Italy the average stock of human capital p.c. is low and the diffusion of technology is very effective, whereas Sweden has the highest average human capital endowment p.c. and at the same time the role of technology diffusion is small.
- The average effect of technology diffusion for the analyzed group of countries is an increase of the growth rate of technology by ca. 2 percent for a country with technology level equal to half of the level of the technology of the technological leader, provided that the R&D expenditures are equal.
- Subsidies to research activity can widely increase the growth rates of technology.
- Apart from technology diffusion, also work on the implementation of the transferred technologies is needed – these technologies have to be adjusted to the requirements of a given country.
- The analyzed countries have different sensitivities with respect to technological backwardness. This sensitivity is rather low for the developing countries and rather high in the developed countries. It results from the fact that technological progress in the former consists mainly in the flow of technology from the technologically advanced firms (which can be interpreted as firms with international capital) to the firms with lower technology levels. In the developed countries, in turn, the levels of technology are similar among different firms and technological progress results more from own research activity or from direct transfer of the most advanced technologies in the world.
- The richest countries have the highest discount rates and the poorest countries tend to have much lower discount rates. For the latter, postponing current consumption in order to increase investments, which in turn augment future consumption, makes it possible to achieve higher rates of growth.

We can also draw conclusions about the construction of economic growth models which include technological progress and human capital.

- To explain the observed cross-country differences, one needs both the effects of human capital and technological progress.
- The traditional approach, in which the growth rate of technology depends solely on the level of R&D expenditures, does not seem to be plausible – it contradicts the empirical observation that the technology of countries with low levels of these expenditures can also grow fast. Thus, it is essential to take the phenomenon of technology diffusion into account.
- The assumption that one country is the technological leader and the rest of countries profit from the technology of the leader is rather too simplified. Such an approach could be appropriate if the group of countries that receive the frontier technology consisted only of the most technologically underdeveloped ones. For the group of the OECD countries, one should apply a model with technology flows between all the countries and not only between the leader and other countries.

- The existence of countries with negative diffusion coefficients seems to suggest that these countries are the suppliers of technology, rather than receivers and/or R&D expenditures are allocated inefficiently. Hence, a measure of efficiency of R&D expenditures should be introduced.
- The processes of technology diffusion can be effectively described with simulational models.
- There are close relations between simulational and neoclassical models. When the number of agents is large or they are close to identical (the parameters that describe them are similar), the simulational models reproduce the results of the neoclassical models. When, in turn, the number of agents is smaller or their diversity is large, we observe big fluctuations and clear cyclical behaviour. Such behaviour is an inherent property of the real-world economies.
- The amplitudes of fluctuations in the simulational models are, for realistic parameter values, similar to the real-world amplitudes. Hence, this class of models can be regarded as the right tool to describe the dynamics of the real-world economies.

Before we finish, let us point to the possible directions of further research on the role of human capital and technological progress for economic growth:

- the construction of more complex models of human capital and technological progress, taking into account technology diffusion and different quality of schooling in different countries,
- the introduction of a measure of efficiency of R&D expenditures in the models of technological progress – this would also make it possible to more appropriately assess the role of technology diffusion,
- the introduction of a more complex mechanism of technology diffusion to take into account technology flows between all countries,
- the implementation of more complex rules of behaviour of agents in simulational models,
- the introduction of more realistic models of human capital accumulation,
- the construction of simulational models with taxes and government expenditures,
- the introduction of a measure of fluctuations in simulational models and the analysis of fluctuations in the real-world economies.

The answers to the questions about the mechanisms of economic growth are very important from the point of view of economic policy and can help to support economic growth, especially in the less developed countries. However, to find these answers, we need to further investigate the role of human capital and technological progress – the most important determinants of economic growth in the long-run.

Appendix A

The solution of the Manuelli-Seshadri model

In this Appendix, we present the full solution of the Manuelli-Seshadri model, which was formulated in [59].

The income maximization problem of the representative individual has the following form:

$$\max \int_6^R e^{-r(a-6)} \left(wh(a)(1-n(a)) - x(a)\right) da - x_E, \tag{A.1}$$

subject to the human capital dynamics equation:

$$\dot{h}(a) = z_h \left(n(a)h(a)\right)^{\gamma_1} x(a)^{\gamma_2} - \delta_h h(a), \qquad a \in [6, R), \tag{A.2}$$

and the technology of early human capital formation $h(6) \equiv h_E$:

$$h(6) \equiv h_E = h_B x_E^{\nu}. \tag{A.3}$$

Human capital resource $h(a)$ is the state variable and the control variables are the fraction of time allocated to human capital accumulation $n(a)$ and the expenditures on market goods devoted to human capital formation $x(a)$.

The hamiltonian of the problem (A.1)-(A.3) reads:

$$H = wh(a)\left(1-n(a)\right) - x(a) + q(a)\left(z_h \left(n(a)h(a)\right)^{\gamma_1} x(a)^{\gamma_2} - \delta_h h(a)\right), \tag{A.4}$$

where $q(a)$ is the costate variable. The first-order conditions are:

$$\frac{\partial H}{\partial n(a)} \geq 0^1, \tag{A.5}$$

$$\frac{\partial H}{\partial x(a)} = 0, \tag{A.6}$$

$$\dot{q} = -\frac{\partial H}{\partial h(a)} + rq, \tag{A.7}$$

$$\dot{h} = \frac{\partial H}{\partial q(a)}, \tag{A.8}$$

where $a \in [6, R)$, with the transversality condition:

$$q(R) = 0. \tag{A.9}$$

We obtain:

$$wh(a)n(a) \geq \gamma_1 z_h q(a) \left(n(a)h(a)\right)^{\gamma_1} x(a)^{\gamma_2\, 2}, \tag{A.10}$$

$$x(a) = \gamma_2 z_h q(a) \left(n(a)h(a)\right)^{\gamma_1} x(a)^{\gamma_2}, \tag{A.11}$$

$$\dot{q}(a) = rq(a) - q(a)\left(\gamma_1 z_h \left(n(a)h(a)\right)^{\gamma_1} x(a)^{\gamma_2} h(a)^{-1} - \delta_h\right) - w\left(1-n(a)\right), \tag{A.12}$$

[1] This condition holds with equality if n(a)<1.

[2] This condition is satisfied with equality if $n(a) < 1$.

$$\dot{h}(a) = z_h \left(n(a)h(a)\right)^{\gamma_1} x(a)^{\gamma_2} - \delta_h h(a). \tag{A.13}$$

We will first find the expression for the expenditures on market goods devoted to human capital accumulation. Equation (A.11) in the period of the formal education ($n(a) = 1$) takes, after division by x^{γ_2}, the following form:

$$x = \left(qh^{\gamma_1}\right)^{\frac{1}{1-\gamma_2}} \left(\gamma_2 z_h\right)^{\frac{1}{1-\gamma_2}}. \tag{A.14}$$

Let us begin with the expression in the first parentheses. Its growth rate reads:

$$\frac{\dot{\left(qh^{\gamma_1}\right)}}{qh^{\gamma_1}} = \left(\frac{\dot{q}}{q} + \gamma_1 \frac{\dot{h}}{h}\right). \tag{A.15}$$

Substituting now into this equation the expression for the growth rates of q (A.12) and h (A.13), in which we set $n = 1$, we obtain:

$$\frac{\dot{\left(qh^{\gamma_1}\right)}}{qh^{\gamma_1}} = r + \delta_h(1 - \gamma_1). \tag{A.16}$$

The solution to this equation is:

$$qh^{\gamma_1} = q_E h_E^{\gamma_1} e^{(r+\delta_h(1-\gamma_1))(a-6)}, \tag{A.17}$$

with $q_E \equiv q(6)$ and $h_E \equiv h(6)$. Substituting now expression (A.17) into (A.14), we obtain the following equation for the expenditures during the schooling period:

$$x(a) = \left(q_E h_E^{\gamma_1} \gamma_2 z_h\right)^{\frac{1}{1-\gamma_2}} e^{\frac{r+\delta_h(1-\gamma_1)}{1-\gamma_2}(a-6)}. \tag{A.18}$$

To find the analogous expression for the working period, we have to first find $q(a)$. We substitute (A.10) with equality (since $n < 1$) into (A.12) and we obtain:

$$\dot{q} = q(r + \delta_h) - w. \tag{A.19}$$

The above differential equation can be solved with the Laplace transform method. Let us denote the Laplace transform of the function $q(a)$ as $\mathscr{L}\{q(a)\} = Y(s)$. Then:

$$sY(s) = (r + \delta_h)Y(s) - \frac{w}{s}. \tag{A.20}$$

After rearrangements, we obtain:

$$Y(s) = \frac{w}{s\left((r + \delta_h) - s\right)}. \tag{A.21}$$

Taking now the inverse Laplace transform, we have:

$$q(a) = \frac{w}{r + \delta_h}\left(1 - e^{-(r+\delta_h)(R-a)}\right). \tag{A.22}$$

Substituting this equation into (A.11), we obtain:

$$x^{1-\gamma_2} = \frac{w}{r + \delta_h}\left(1 - e^{-(r+\delta_h)(R-a)}\right) \gamma_2 z_h (nh)^{\gamma_1}. \tag{A.23}$$

Dividing (A.10) by (A.11), we have:

$$nh = \frac{\gamma_1}{\gamma_2 w} x. \tag{A.24}$$

Using this relationship in (A.23), we obtain:

$$x^{1-\gamma_2} = \frac{w}{r+\delta_h}\left(1-e^{-(r+\delta_h)(R-a)}\right)\gamma_2 z_h \left(\frac{\gamma_1}{\gamma_2 w}x\right)^{\gamma_1} x^{\gamma_1}. \tag{A.25}$$

After rearrangements, we obtain the final expression for the working period:

$$x(a) = \frac{\gamma_2 w}{r+\delta_h}\left(\frac{\gamma_2^{\gamma_2}\gamma_1^{\gamma_1} z_h w^{\gamma_2}}{(r+\delta_h)^{\gamma}}\right)^{\frac{1}{1-\gamma}}\left(1-e^{-(r+\delta_h)(R-a)}\right)^{\frac{1}{1-\gamma}}, \tag{A.26}$$

where $\gamma = \gamma_1 + \gamma_2$.

We will now find the optimal paths of human capital. Substituting expression (A.18) to the human capital dynamics equation (A.13), we obtain a non-linear first-order differential equation:

$$\dot{h} = (q_E^{\gamma_2} h_E^{\gamma_1\gamma_2}\gamma_2^{\gamma_2} z_h)^{\frac{1}{1-\gamma_2}} e^{\frac{\gamma_2(r+\delta_h(1-\gamma_1))}{1-\gamma_2}(a-6)} h^{\gamma_1} - \delta_h h. \tag{A.27}$$

Let us denote:

$$A \equiv (q_E^{\gamma_2} h_E^{\gamma_1\gamma_2}\gamma_2^{\gamma_2} z_h)^{\frac{1}{1-\gamma_2}}, \tag{A.28}$$

$$B \equiv \frac{\gamma_2(r+\delta_h(1-\gamma_1))}{1-\gamma_2}. \tag{A.29}$$

We thus obtain a differential equation of the Bernoulli equation form:

$$\dot{h} = Ae^{B(a-6)}h^{\gamma_1} - \delta_h h. \tag{A.30}$$

We can obtain a linear differential equation with the following substitution:

$$y = h^{1-\gamma_1}. \tag{A.31}$$

Thus:

$$\dot{h} = \frac{h^{\gamma_1}}{1-\gamma_1}\dot{y}. \tag{A.32}$$

After substituting (A.31) and (A.32) into (A.30), we obtain a linear equation:

$$\frac{\dot{y}}{1-\gamma_1} = Ae^{Ba} - \delta_h y, \tag{A.33}$$

where we have also set $(a-6) \to a$. This equation can be again solved by the Laplace transform. We take:

$$\mathscr{L}\{y(a)\} = Y(s). \tag{A.34}$$

After Laplace transform of the whole equation (A.33), we obtain:

$$\frac{1}{1-\gamma_1}\left(sY(s) - h_E^{1-\gamma_1}\right) = \frac{A}{s-B} - \delta_h Y(s), \tag{A.35}$$

where $h_E^{1-\gamma_1} = y(0)$. After rearrangements, we obtain an expression for $Y(s)$:

$$Y(s) = \frac{h_E^{1-\gamma_1}}{s+\delta_h(1-\gamma_1)} + \frac{A(1-\gamma_1)}{(s-B)(s+\delta_h(1-\gamma_1))}. \tag{A.36}$$

Taking now the inverse Laplace transform, we have:

$$y(a) = h_E^{1-\gamma_1}e^{-\delta_h(1-\gamma_1)a} + \frac{A(1-\gamma_1)}{B+\delta_h(1-\gamma_1)}\left(e^{Ba} - e^{-\delta_h(1-\gamma_1)a}\right). \tag{A.37}$$

Returning to the variable $h(a)$ and substituting explicit expressions for A

and B, and finally setting $a \to (a-6)$, we obtain the final expression for $h(a)$ in the schooling period:

$$\begin{aligned} h(a) &= h_E e^{-\delta_h(a-6)} \Bigg(1 + \left(h_E^{-(1-\gamma)} q_E^{\gamma_2} \gamma_2^{\gamma_2} z_h \right)^{\frac{1}{1-\gamma_2}} \\ &\times \frac{(1-\gamma_1)(1-\gamma_2)}{\gamma_2 r + \delta_h(1-\gamma_1)} \left(e^{\frac{\gamma_2 r + \delta_h(1-\gamma_1)}{1-\gamma_2}(a-6)} - 1 \right) \Bigg)^{\frac{1}{1-\gamma_1}}. \end{aligned} \quad \text{(A.38)}$$

To find the analogous expression for the working period, we substitute the expressions for $x(a)$ (A.26) and $q(a)$ (A.22):

$$\begin{aligned} \dot{h} &= z_h (nh)^{\gamma_1} x^{\gamma_2} - \delta_h h = \frac{x}{q\gamma_2} - \delta_h h = \\ &= \left(\frac{\gamma_2^{\gamma_2} \gamma_1^{\gamma_1} z_h w^{\gamma_2}}{(r+\delta_h)^{\gamma}} \right)^{\frac{1}{1-\gamma}} \left(1 - e^{-(r+\delta_h)(R-a)} \right)^{\frac{\gamma}{1-\gamma}} - \delta_h h. \end{aligned} \quad \text{(A.39)}$$

Let us denote:

$$C \equiv \left(\frac{\gamma_2^{\gamma_2} \gamma_1^{\gamma_1} z_h w^{\gamma_2}}{(r+\delta_h)^{\gamma}} \right)^{\frac{1}{1-\gamma}}, \quad \text{(A.40)}$$

$$m(a) = \left(1 - e^{-(r+\delta_h)(R-a)} \right). \quad \text{(A.41)}$$

The considered differential equation has thus the form:

$$\dot{h}(a) = C m(a)^{\frac{\gamma}{1-\gamma}} - \delta_h h(a). \quad \text{(A.42)}$$

We look for the solution in the interval $a \in [6+s, R]$. The form of $\dot{h}$ implies that the solution can be written in the form:

$$h(a) = e^{-\delta_h(a-6-s)} \left(h(6+s) + C g(a) \right), \quad \text{(A.43)}$$

where $g(a)$ is the function we are looking for. It satisfies $g(6+s) = 0$. We thus have:

$$\dot{h}(a) = -\delta_h h(a) + e^{\delta_h(a-6-s)} C \dot{g}(a). \quad \text{(A.44)}$$

Comparing eqs. (A.42) and (A.44), we obtain:

$$\dot{g}(a) = e^{\delta_h(a-6-s)} m(a)^{\frac{\gamma}{1-\gamma}}. \quad \text{(A.45)}$$

The form of the function $m(a)$ implies that the solution to this equation has to be of the form:

$$g(a) = D \int_{const}^{u(a)} f(x) dx, \quad \text{(A.46)}$$

where D and *const* are constants and f and u are the functions we are searching for. The derivative of the function $g(a)$ has the form:

$$\dot{g}(a) = D f\left(u(a)\right) \dot{u}(a). \quad \text{(A.47)}$$

The upper limit of the integral (A.46) has to lead to the function $m(a)$, hence we have to assume:

$$u(a) = e^{\delta_h(a-R)}. \quad \text{(A.48)}$$

Thus:

$$\dot{g}(a) = \underbrace{\underbrace{\frac{1}{\delta_h} e^{-\delta_h(6+s-R)}}_{D} \underbrace{\delta_h e^{\delta_h(a-R)}}_{\dot{u}(a)}}_{e^{\delta_h(a-6-s)}} m(a)^{\frac{\gamma}{1-\gamma}}. \tag{A.49}$$

The function f in (A.46) thus has to have the form:

$$f(x) = \left(1 - x^{\frac{r+\delta_h}{\delta_h}}\right)^{\frac{\gamma}{1-\gamma}}. \tag{A.50}$$

The exponent $(r + \delta_h)/\delta_h$ is taken in order to eliminate terms of the a-dependent form e^{ra} in the function $g(a)$. We obtain:

$$g(a) = \frac{1}{\delta_h} e^{-\delta_h(6+s-R)} \int_{e^{\delta_h(6+s-R)}}^{e^{\delta_h(a-R)}} dx \left(1 - x^{\frac{r+\delta_h}{\delta_h}}\right)^{\frac{\gamma}{1-\gamma}}, \tag{A.51}$$

where the lower limit of integration results from the condition $g(6 + s) = 0$. Finally, after substituting the expression for the constant C, we have:

$$\begin{aligned} h(a) &= e^{-\delta_h(a-6-s)} \left(h(6+s) + \frac{1}{\delta_h} \left(\frac{\gamma_2^{\gamma_2} \gamma_1^{\gamma_1} z_h w^{\gamma_2}}{(r+\delta_h)^{\gamma}} \right)^{\frac{1}{1-\gamma}} \right. \\ &\times \left. e^{-\delta_h(6+s-R)} \int_{e^{\delta_h(6+s-R)}}^{e^{\delta_h(a-R)}} dx \left(1 - x^{\frac{r+\delta_h}{\delta_h}}\right)^{\frac{\gamma}{1-\gamma}} \right). \end{aligned} \tag{A.52}$$

We will now find the expression for the fraction of time allocated to human capital formation. In the schooling period $n(a) = 1$, by definition. For $a \in [6 + s, R)$, equation (A.24) is satisfied. Dividing this equation by h and substituting the expressions for x and h, we obtain:

$$n(a) = \frac{m(a)^{\frac{1}{1-\gamma}}}{e^{-\delta_h(a-s-6)} m(6+s)^{\frac{1}{1-\gamma}} + \frac{(r+\delta_h)e^{\delta_h(R-a)}}{\gamma_1 \delta_h} \int_{x_{6+s}}^{x_a} dx \left(1 - x^{\frac{r+\delta_h}{\delta_h}}\right)^{\frac{\gamma}{1-\gamma}}}, \tag{A.53}$$

where $x_a = e^{\delta_h(a-R)}$, $x_{6+s} = e^{\delta_h(6+s-R)}$.

The solution will be complete, if we find the expressions for h_E and q_E. Let us consider the case $a = 6 + s$. Eq. (A.11) implies that:

$$x(a)^{\gamma_2} = (\gamma_2 z_h (n(a)h(a))^{\gamma_1} q(a))^{\frac{\gamma_2}{1-\gamma_2}}. \tag{A.54}$$

Substituting this expression into eq. (A.10) and taking $n(6 + s) = 1$ into account, we can simplify to the following relationship between $h(6 + s)$ and $q(6 + s)$:

$$h(6+s) = \gamma_1^{\frac{1-\gamma_2}{1-\gamma}} \gamma_2^{\frac{\gamma_2}{1-\gamma}} z_h^{\frac{1}{1-\gamma}} w^{\frac{\gamma_2-1}{1-\gamma}} q(6+s)^{\frac{1}{1-\gamma}}. \tag{A.55}$$

We substitute this expression to eq. (A.17) and obtain:

$$q(6+s) = \frac{w^{\gamma_1}}{\gamma_1^{\gamma_1} \gamma_2^{\frac{\gamma_1\gamma_2}{1-\gamma_2}} z_h^{\frac{\gamma_1}{1-\gamma_2}}} e^{\frac{(r+\delta_h(1-\gamma_1))(1-\gamma)s}{1-\gamma_2}} q_E^{\frac{1-\gamma}{1-\gamma_2}} h_E^{\frac{\gamma_1(1-\gamma)}{1-\gamma_2}}. \tag{A.56}$$

From eq. (A.22), we have:

$$q(6+s) = \frac{w}{r+\delta_h} m(6+s). \tag{A.57}$$

Combining the two equations given above, we obtain the following relationship between q_E and h_E:

$$q_E = \left(\frac{w^{(1-\gamma_1)(1-\gamma_2)} \gamma_1^{\gamma_1(1-\gamma_2)} \gamma_2^{\gamma_1\gamma_2} z_h^{\gamma_1} m(6+s)^{1-\gamma_2}}{(r+\delta_h)^{1-\gamma_2}} \right)^{\frac{1}{1-\gamma}} e^{-(r+\delta_h(1-\gamma_1))s} h_E^{-\gamma_1}. \tag{A.58}$$

The variable q_E represents the costate variable for age $a = 6$. The parents choose for their children such level of pre-school expenditures x_E that the income from such investment $q_E h_E$ is maximal. If $\pi = q_E h_E - x_E$, the optimal level of expenditures x_E satisfies:

$$\frac{\partial \pi}{\partial x_E} = 0. \tag{A.59}$$

Substituting $h_E = h_B x_E^v$ in the above equation, we obtain:

$$q_E h_B v x_E^{v-1} = 1. \tag{A.60}$$

Hence,

$$h_E = (v^v h_B)^{\frac{1}{1-v}} q_E^{\frac{v}{1-v}}. \tag{A.61}$$

It is easy to check that it corresponds to the maximum of the function π (since $v < 1$). Substituting this equation into (A.58), we obtain:

$$q_E = \frac{e^{-(1-v)(r+\delta_h(1-\gamma_1))s}}{h_B v^v} \left(\frac{w^{(1-\gamma_1)(1-\gamma_2)} \gamma_1^{\gamma_1(1-\gamma_2)} \gamma_2^{\gamma_1\gamma_2} z_h^{\gamma_1} m(6+s)^{1-\gamma_2}}{(r+\delta_h)^{1-\gamma_2}} \right)^{\frac{1-v}{1-\gamma}}. \tag{A.62}$$

Substituting (A.58) into (A.61), we have:

$$h_E = \left(h_B v^v \left(\frac{w^{(1-\gamma_1)(1-\gamma_2)} \gamma_1^{\gamma_1(1-\gamma_2)} \gamma_2^{\gamma_1\gamma_2} z_h^{\gamma_1}}{(r+\delta_h)^{1-\gamma_2} m(6+s)^{\gamma_2-1}} \right)^{\frac{v}{1-\gamma}} e^{-v(r+\delta_h(1-\gamma_1))s} \right)^{\frac{1}{1-v(1-\gamma_1)}}, \tag{A.63}$$

In this way, we obtained a complete set of expressions for $x(a)$, $h(a)$ and $n(a)$, dependent only on the model parameter values: s, w, r, δ_h, z_h, γ_1, γ_2, R, h_B, v.

Appendix B

The method of estimating the technological parameters A and g

The level of technology A is found from the Cobb-Douglas production function:

$$y = A^{1-\theta} k^{\theta}, \tag{B.1}$$

where y denotes the empirical GDP p.c., k – physical capital p.c., $\theta = 0.315$ for every country, according to Manuelli and Seshadri [59]. Hence,

$$A = \left(yk^{\theta}\right)^{\frac{1}{1-\theta}}. \tag{B.2}$$

The values of y and k for the OECD countries for the years 1960-1999, taken from [57].

The rate of technological progress was found as the geometrical mean of the growth rates of A for the last 10 years, i.e. the period 1989-1999. The growth rate of A for an arbitrary period t is calculated from the following equation:

$$g_t = \frac{A_t - A_{t-1}}{A_{t-1}}, \tag{B.3}$$

where A_t denotes the level of technology for the year t.

Country symbols:
AUS – Australia,
AUT – Austria,
BEL – Belgium,
CAN – Canada,
CZE – the Czech Republic,
DEN – Denmark,
FIN – Finland,
FRA – France,
GER – Germany,
GRE – Greece,
HUN – Hungary,
IRL – Ireland,
ISL – Iceland,
ITA – Italy,
JPN – Japan,
KOR – South Korea,
LUX – Luxembourg,
MEX – Mexico,
NED – The Netherlands,
NZL – New Zealand,
NOR – Norway,

POL – Poland,
POR – Portugal,
SPA – Spain,
SWE – Sweden,
SWI – Switzerland,
TUR – Turkey,
UK – United Kingdom,
USA – United States.

Appendix C

The data on R&D expenditures and the number of researchers in the OECD countries

In the tables below, we show the data on R&D expenditures and the number of researchers per 1000 employed in the OECD countries.

Under the notion of research and development we understand all systematic activities aiming at increasing knowledge and using it in practice. The measure of R&D expenditures that we use is *Gross domestic Expenditure on R&D* (GERD), which consists of all research expenditures of domestic firms, universities, research institutes, government laboratories etc. The foreign expenditures of domestic subjects are not included.

Under the notion of a researcher we understand all workers of domestic research entities, which create new knowledge and also new products, processes and methods. Project managers are also included.

The missing data in the following tables were supplemented with interpolated values.

Table C.1: **R&D expenditures in the OECD countries as a fraction of GDP in the period 1981-1990**

Country	1981	1982	1983	1984	1985	1986	1987	1988	1989	1990
AUS	0.95	1.02	1.10	1.17	1.24	1.29	1.33	1.38	1.43	1.47
AUT	1.13	1.16	1.19	1.21	1.24	1.28	1.32	1.36	1.39	1.43
BEL	–	–	–	–	1.62	1.62	1.62	1.62	1.62	1.62
CAN	1.24	1.29	1.34	1.39	1.44	1.47	1.49	1.52	1.55	1.57
CZE	–	–	–	–	–	–	–	–	–	–
DEN	1.06	1.10	1.14	1.17	1.21	1.28	1.35	1.43	1.50	1.57
FIN	1.17	1.27	1.36	1.46	1.55	1.63	1.71	1.79	1.87	1.95
FRA	1.93	2.00	2.08	2.15	2.22	2.25	2.27	2.30	2.32	2.35
GER	2.43	2.49	2.56	2.62	2.68	2.66	2.63	2.61	2.58	2.56
GRE	0.17	0.20	0.22	0.25	0.27	0.29	0.30	0.32	0.33	0.35
HUN	–	–	–	–	–	–	–	–	–	–
IRL	0.68	0.70	0.73	0.75	0.77	0.80	0.82	0.85	0.88	0.90
ISL	0.64	0.67	0.69	0.72	0.74	0.81	0.89	0.96	1.03	1.11
ITA	0.88	0.94	1.00	1.06	1.12	1.14	1.16	1.18	1.19	1.21
JPN	2.11	2.22	2.33	2.43	2.54	2.58	2.61	2.65	2.68	2.72
KOR	–	–	–	–	–	–	–	–	–	–
MEX	–	–	–	–	–	–	–	–	–	–
NED	1.79	1.84	1.89	1.94	1.99	1.99	1.98	1.98	1.98	1.97
NZL	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99
NOR	1.17	1.25	1.33	1.40	1.48	1.51	1.53	1.56	1.59	1.61
POL	–	–	–	–	–	–	–	–	–	–
POR	0.30	0.32	0.34	0.36	0.38	0.42	0.46	0.50	0.53	0.57
SPA	0.41	0.44	0.47	0.50	0.53	0.58	0.63	0.69	0.74	0.79
SWE	2.17	2.31	2.44	2.58	2.71	2.71	2.71	2.71	2.70	2.70
SWI	2.18	2.34	2.50	2.66	2.82	2.79	2.77	2.74	2.71	2.69
TUR	–	–	–	–	–	–	–	–	–	–
UK	2.38	2.35	2.31	2.28	2.24	2.21	2.18	2.16	2.13	2.10
USA	2.34	2.45	2.55	2.66	2.76	2.75	2.75	2.74	2.73	2.73

Source: [81].

Table C.2: **R&D expenditures in the OECD countries as a fraction of GDP in the period 1991-1999**

Country	1991	1992	1993	1994	1995	1996	1997	1998	1999
AUS	1.52	1.55	1.58	1.60	1.63	1.66	1.59	1.51	1.52
AUT	1.47	1.49	1.52	1.54	1.56	1.60	1.71	1.78	1.85
BEL	1.62	1.65	1.67	1.70	1.72	1.80	1.87	1.90	1.96
CAN	1.60	1.63	1.66	1.69	1.72	1.68	1.68	1.79	1.81
CZE	2.02	1.77	1.52	1.26	1.01	1.04	1.16	1.24	1.24
DEN	1.64	1.69	1.74	1.79	1.84	1.85	1.94	2.06	2.19
FIN	2.03	2.09	2.16	2.22	2.28	2.54	2.71	2.88	3.23
FRA	2.37	2.36	2.34	2.33	2.31	2.30	2.22	2.17	2.18
GER	2.53	2.46	2.40	2.33	2.26	2.26	2.29	2.31	2.44
GRE	0.36	0.39	0.43	0.46	0.49	0.50	0.51	0.59	0.67
HUN	1.06	0.98	0.90	0.81	0.73	0.65	0.72	0.68	0.69
IRL	0.93	1.02	1.11	1.19	1.28	1.32	1.29	1.25	1.22
ISL	1.18	1.28	1.38	1.47	1.57	1.73	1.88	2.07	2.39
ITA	1.23	1.17	1.12	1.06	1.00	1.01	1.05	1.07	1.04
JPN	2.75	2.74	2.72	2.71	2.69	2.77	2.83	2.94	2.94
KOR	1.92	2.07	2.21	2.36	2.50	2.60	2.69	2.55	2.47
MEX	0.22	0.24	0.27	0.29	0.31	0.31	0.34	0.38	0.43
NED	1.97	1.98	1.98	1.99	1.99	2.01	2.04	1.94	2.02
NZL	0.98	0.98	0.97	0.97	0.96	1.04	1.11	1.07	1.03
NOR	1.64	1.66	1.67	1.69	1.70	1.67	1.64	1.65	1.65
POL	0.69	0.69	0.69	0.69	0.69	0.71	0.71	0.72	0.75
POR	0.61	0.60	0.59	0.58	0.57	0.60	0.62	0.69	0.75
SPA	0.84	0.83	0.83	0.82	0.81	0.83	0.82	0.89	0.88
SWE	2.70	2.86	3.03	3.19	3.35	3.45	3.54	3.60	3.65
SWI	2.66	2.67	2.69	2.70	2.72	2.73	2.71	2.68	2.66
TUR	0.53	0.49	0.46	0.42	0.38	0.45	0.49	0.50	0.63
UK	2.07	2.04	2.01	1.98	1.95	1.88	1.81	1.80	1.88
USA	2.72	2.67	2.62	2.56	2.51	2.55	2.58	2.60	2.65

Source: [81].

Table C.3: **The number of researchers per 1000 employed in the OECD countries 1981-1990**

Country	1981	1982	1983	1984	1985	1986	1987	1988	1989	1990
AUS	3.62	3.83	4.05	4.26	4.27	4.89	4.99	5.21	5.34	5.47
AUT	1.78	1.85	1.92	1.98	2.05	2.12	2.18	2.25	2.31	2.55
BEL	3.44	3.67	3.72	3.85	4.07	4.30	4.37	4.44	4.63	4.66
CAN	3.52	3.97	4.08	4.25	4.43	4.70	4.72	4.79	4.79	4.90
CZE	1.88	1.97	2.05	2.09	2.07	2.13	2.20	2.21	2.22	2.27
DEN	2.83	3.01	3.17	3.26	3.34	3.46	3.69	3.87	4.11	4.35
FIN	3.89	3.89	3.89	4.00	4.11	4.22	4.33	4.75	5.16	5.58
FRA	3.87	4.06	4.18	4.48	4.67	4.77	4.95	5.16	5.32	5.42
GER	4.68	4.84	4.99	5.17	5.35	5.59	5.83	5.92	6.01	6.14
GRE	1.26	1.32	1.38	1.40	1.39	1.43	1.48	1.48	1.49	1.61
HUN	2.61	2.74	2.86	2.91	2.87	2.97	3.06	3.08	3.09	3.15
IRL	1.84	1.99	2.18	2.40	2.52	2.68	2.99	3.21	3.67	3.96
ISL	7.93	8.32	8.67	8.84	8.72	9.00	9.28	9.33	9.38	9.56
ITA	2.44	2.65	2.94	2.89	2.94	3.11	3.23	3.39	3.42	3.44
JPN	5.26	5.39	5.74	5.89	6.24	6.41	6.75	6.98	7.24	7.44
KOR	4.01	4.21	4.38	4.47	4.41	4.55	4.69	4.72	4.74	4.83
MEX	0.35	0.37	0.39	0.39	0.39	0.40	0.41	0.42	0.42	0.43

NED	3.35	3.75	4.15	4.22	4.28	4.19	4.10	4.10	4.10	4.24
NZL	3.36	3.52	3.67	3.74	3.69	3.81	3.93	3.95	3.97	4.12
NOR	3.81	3.94	4.22	4.54	4.77	5.07	5.36	5.62	5.87	6.24
POL	2.44	2.56	2.67	2.72	2.68	2.77	2.86	2.87	2.89	2.94
POR	0.97	1.02	1.04	1.10	1.27	1.42	1.49	1.55	1.65	1.74
SPA	1.53	1.50	1.56	1.68	1.78	1.99	2.05	2.33	2.37	2.62
SWE	4.20	4.35	4.50	4.77	5.04	5.10	5.16	5.41	5.65	5.77
SWI	3.66	3.84	4.00	4.08	4.02	4.15	4.17	4.18	4.20	4.27
TUR	0.49	0.51	0.54	0.55	0.54	0.56	0.57	0.58	0.58	0.59
UK	4.89	5.02	4.99	4.96	4.98	5.08	4.97	4.90	4.63	4.61
USA	6.61	6.94	7.23	7.37	7.27	7.51	7.74	7.78	7.82	7.97

Source: [82].

Table C.4: **The number of researchers per 1000 employed in the OECD countries 1991-1999**

Country	1991	1992	1993	1994	1995	1996	1997	1998	1999
AUS	6.11	6.75	6.86	6.96	7.14	7.31	7.30	7.28	7.48
AUT	2.79	3.03	3.27	3.55	3.83	4.10	4.38	4.66	4.79
BEL	4.69	5.09	5.48	5.92	6.07	6.53	6.76	6.97	7.41
CAN	5.14	5.51	5.71	6.38	6.38	6.55	6.62	6.61	6.67
CZE	2.31	2.32	2.34	2.33	2.31	2.51	2.43	2.49	2.80
DEN	4.58	4.95	5.34	5.71	6.07	6.29	6.52	6.70	6.88
FIN	5.99	6.72	7.44	7.82	8.20	10.25	12.30	13.85	14.51
FRA	5.67	6.23	6.50	6.64	6.67	6.80	6.77	6.71	6.78
GER	6.26	6.23	6.21	6.18	6.15	6.14	6.29	6.27	6.63
GRE	1.72	1.94	2.16	2.35	2.54	2.72	2.90	3.32	3.74
HUN	3.21	3.02	3.09	3.13	2.90	2.89	3.09	3.19	3.31
IRL	4.44	4.75	4.09	4.31	4.49	4.82	5.02	5.06	4.86
ISL	9.74	9.81	9.88	9.82	9.75	10.15	10.54	10.84	11.14
ITA	3.27	3.25	3.33	3.44	3.43	3.45	2.96	2.91	2.87
JPN	7.49	7.71	7.91	8.12	8.27	9.21	9.24	9.71	9.88
KOR	4.92	4.96	5.00	4.96	4.93	4.77	4.84	4.65	4.94
MEX	0.43	0.44	0.44	0.52	0.58	0.57	0.58	0.54	0.56
NED	4.37	4.51	4.64	4.89	4.85	4.86	5.04	5.05	5.08
NZL	3.96	5.06	5.26	5.00	4.74	5.47	6.20	6.21	6.22
NOR	6.61	6.92	7.23	7.39	7.54	7.71	7.88	7.93	7.97
POL	3.00	3.02	3.04	3.02	3.18	3.27	3.41	3.41	3.56
POR	1.88	2.05	2.19	2.29	2.59	2.77	2.95	3.09	3.25
SPA	2.79	2.90	3.11	3.45	3.35	3.60	3.65	3.93	3.88
SWE	5.89	6.53	7.17	7.69	8.21	8.70	9.18	9.39	9.59
SWI	4.34	4.41	4.70	5.00	5.29	5.58	5.80	5.96	6.13
TUR	0.62	0.65	0.70	0.73	0.78	0.86	0.90	0.88	0.92
UK	4.57	4.71	4.82	4.89	5.26	5.18	5.13	5.49	5.64
USA	8.12	8.18	8.24	8.19	8.13	8.46	8.79	9.04	9.29

Source: [82].

Table C.5: **Research human capital resources in the OECD countries for 1981-1990**

Country	1981	1982	1983	1984	1985	1986	1987	1988	1989	1990
AUS	341	365	389	414	419	485	500	527	545	564
AUT	127	133	140	146	152	159	165	172	179	199
BEL	307	331	339	354	378	403	414	425	447	455
CAN	365	416	432	455	479	513	520	533	538	556
CZE	167	177	187	192	191	199	208	211	214	220
DEN	268	288	306	318	329	345	371	393	422	451
FIN	396	400	404	420	436	452	468	518	569	621
FRA	259	274	285	309	325	335	351	370	385	396

GER	451	471	491	513	537	566	596	612	627	647
GRE	94	99	105	108	107	112	117	118	120	131
HUN	176	187	197	203	202	211	219	223	226	233
IRL	157	171	189	210	223	240	270	293	338	368
ISL	604	640	673	693	691	720	750	762	773	796
ITA	157	173	193	192	197	211	221	234	239	243
JPN	432	447	481	498	533	553	588	614	643	668
KOR	388	412	433	446	444	463	483	490	497	512
MEX	22	23	24	25	25	26	27	27	28	28
NED	330	374	418	428	439	434	429	434	438	457
NZL	331	352	370	381	379	396	412	418	425	445
NOR	408	426	461	501	532	570	610	645	681	731
POL	139	148	155	160	159	166	173	176	178	184
POR	41	43	44	47	55	63	66	70	75	80
SPA	94	94	98	107	114	129	134	154	158	177
SWE	482	505	527	564	602	616	629	666	703	725
SWI	315	334	351	361	360	375	381	386	391	402
TUR	22	23	25	25	25	26	27	28	28	29
UK	433	449	450	452	458	472	467	465	444	446
USA	668	709	746	768	765	797	831	843	856	881

Source: own computations based on Tabs. C.3, C.4, 2.9 and 2.11.

Table C.6: **Research human capital resources in the OECD countries for 1991-1999**

Country	1991	1992	1993	1994	1995	1996	1997	1998	1999
AUS	637	710	729	747	774	800	807	813	844
AUT	220	241	263	288	314	340	367	394	409
BEL	462	506	551	601	622	676	707	736	791
CAN	589	638	668	754	761	789	806	813	828
CZE	227	231	235	236	236	259	254	262	298
DEN	479	523	570	615	661	692	724	752	780
FIN	674	763	854	907	960	1213	1470	1671	1769
FRA	419	465	490	505	513	528	531	531	542
GER	666	670	674	678	681	687	711	716	764
GRE	142	161	181	199	217	235	253	293	333
HUN	239	228	235	241	225	227	245	255	267
IRL	417	451	392	417	439	476	501	510	495
ISL	819	833	848	851	853	897	941	978	1015
ITA	233	234	242	252	254	258	224	222	221
JPN	679	706	732	758	780	878	889	944	970
KOR	527	536	545	547	549	536	550	533	572
MEX	29	30	30	36	41	40	42	39	41
NED	476	496	516	549	550	557	583	590	599
NZL	432	558	585	562	538	627	718	726	735
NOR	782	827	873	901	929	959	990	1006	1022
POL	189	192	196	196	209	217	228	231	243
POR	87	96	103	109	125	135	145	153	163
SPA	190	200	216	242	238	258	264	287	287
SWE	747	837	928	1005	1084	1159	1236	1276	1317
SWI	413	423	456	489	523	558	585	608	631
TUR	31	33	35	37	40	45	47	47	50
UK	447	465	480	492	535	532	532	575	597
USA	907	923	939	942	945	993	1042	1082	1123

Source: own computations based on Tabs. C.3, C.4, 2.9 and 2.11.

Appendix D

Random number generation

A very important issue in the context of Monte Carlo simulations is the correct generation of random numbers from a given statistical distribution. In particular, the obtained numbers should be *independent* from each other. This means that the conditional probability of generating any number x_i subject to the fact that the previous number was x_{i-1} should equal the unconditional probability of generation of x_i.

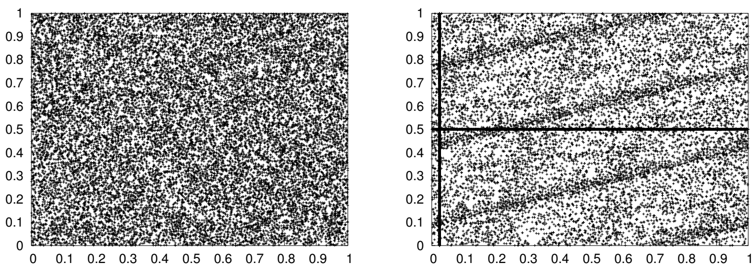
To generate random numbers from the uniform distribution on the interval $(0, 1)$ we use the following algorithm:

1. Choose an irrational number $seed$ and a large integer $mult$.
2. Calculate the fractional part of $seed$ and denote it $SEED$.
3. Calculate $\xi_i = \text{modf}(i * \text{modf}(i * SEED)) * mult + 0.5$, where "modf" denotes the fractional part and the index i numbers the generated random numbers.
4. The i-th generated random number is $x_i = \text{modf}(\xi_i * \text{modf}(\xi_i * SEED))$.

Iterating the algorithm for $i = 1, 2, \ldots$, we obtain a sequence of random numbers. To be precise, these are *pseudo*random numbers, because they result from an algorithm. Thus, if we restart the algorithm from the same values of $seed$ and $mult$, we will obtain the same sequence of numbers as before. This makes it possible to repeat every simulation.

As we have shown, the obtained sequence of numbers is not truly random. However, what is really important from the point of view of a simulation is that every number from a given probability distribution occurs with equal

Figure D.1: **The first-order correlations for different generators of the uniform distribution. (left)** $seed = \sqrt{5}$, **(right)** $seed = 0.1$



frequency. To check the *quality* of pseudorandom numbers, we can calculate the so-called *first-order correlations* between the numbers. We divide the sequence into pairs (x_1, x_2), (x_3, x_4), (x_5, x_6), $\ldots$, which represent the inner points of a unit square. If the pattern of points inside the square is uniform (if no regular patterns occur), we can conclude that the quality of random numbers is sufficient for Monte Carlo simulations.

Fig. D.1 shows the first-order correlations for two different generators of random numbers, which differ only in the value of the parameter *seed* ($mult = 123456789$ in both cases). On the left, we observe no clear pattern, i.e. a uniform distribution of points. This means that the random numbers are sufficiently random. On the right, we can see a clear pattern which means that some pairs of numbers occur more frequently than some other pairs – such random number generator should not be thus used in Monte Carlo simulations.

If we want to generate a number form a uniform distribution on an arbitrary interval (a, b), it is enough to rescale the generated numbers x_i: $x_i \to bx_i + a$.

If we want to obtain a sequence of pseudorandom numbers from a normal distribution with an expected value of $\bar{x}$ and a standard deviation σ, we should use the Box-Muller transformation:

$$y_i = \sqrt{-2 \ln x_i} \cos 2\pi x_i'. \tag{D.1}$$

If the numbers x_i and x_i' come from independent uniform distributions on the interval $(0, 1)$, then the numbers $\sigma y_i + \bar{x}$ will come from the assumed normal distribution.

Figure D.2: **The first-order correlations for different generators of the standardized normal distribution. (left)** $seed = \sqrt{5}$**, (right)** $seed = 0.1$

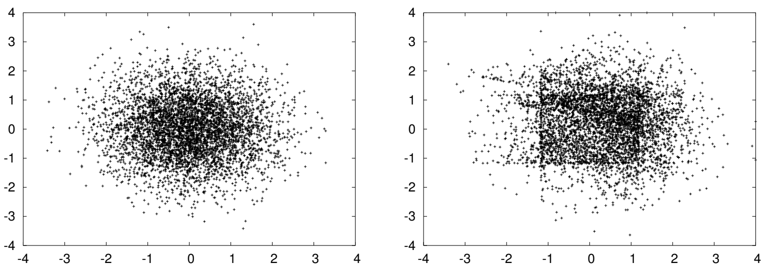


Fig. D.2 shows the first-order correlations for a good random number generator (left panel; we observe a nearly circular shape; inside a unit circle we have approx. 68.3% of points; inside a circle of radius 3σ we observe approx. 99.7% points) and for a bad generator (right panel; we observe clear patterns, which indicate the presence of correlations between subsequent numbers).

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